AD-A010 270

AN INVERSE BOUNDARY-LAYER METHOD FOR COMPRESSIBLE LAMINAR AND TURBULENT FLOWS

Tuncer Cebeci

California State University

Prepared for:

Office of Naval Research

8 April 1975

Department of Mechanical Engineering
California State University at Long Beach
Long Beach, California

April 1975

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER TR-75-1	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER AD-A010 270
4. TITLE (and Subtitle) AN INVERSE BOUNDARY-LAYER METHOD FOR COMPRESSIBLE LAMINAR AND TURBULENT FLOWS		5. TYPE OF REPORT & PERIOD COVERED Final Technical Report Jan 1, 74 - Jan 1, 75
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Tuncer Cebeci		8. CONTRACT OR GRANT NUMBER(s) N00014-74-A-0203-0001 NR 215-234
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mechanical Engineering Dept. California State University at Long Beach Long Beach, California 90840		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Vehicle Warfare Technology Operational Applications Division Office of Naval Research Arlington, Va. 22217		12. REPORT DATE April 8, 1975
		13. NUMBER OF PAGES ~~43~~ 49
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

This document has been approved for public release; its distribution is unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Inverse boundary layers
Laminar flows
Turbulent flows

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This report presents an efficient method for calculating laminar and turbulent boundary-layer flows for standard and inverse boundary-value problems. It is applicable to both incompressible and compressible flows. The standard boundary-layer problem considers the solution of the usual boundary-layer equations for a given external velocity distribution. The inverse problem considers the solution of the governing equations for assigned wall shear or assigned displacement thickness.

PRICES SUBJECT TO CHANGE

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE
S/N 0102-014-6601

Unclassified
SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

AN INVERSE BOUNDARY-LAYER METHOD FOR COMPRESSIBLE LAMINAR AND TURBULENT FLOWS

by

Tuncer Cebeci

Report No. TR-75-1

Prepared under
Contract No. N00014-74-A-0203-0001, NR 215-234
for
THE OFFICE OF NAVAL RESEARCH

Department of Mechanical Engineering
California State University at Long Beach
Long Beach, California

April 1975

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF SYMBOLS

A	Van Driest damping length parameter
c_f	local skin-friction coefficient
C	viscosity-density parameter
f	dimensionless stream function
f'	u/u_e
g	dimensionless total-enthalpy ratio, H/H_e
h	specific enthalpy
H	total enthalpy, or shape factor, δ^*/θ, wherever applicable
K	variable grid parameter
L	modified mixing length
M	Mach number
p	pressure
p^+	dimensionless pressure-gradient parameter
Pr	Prandtl number
Pr_t	turbulent Prandtl number
R_x	Reynolds number, $u_e x/\nu_e$
R_θ	Reynolds number, $u_e \theta/\nu_e$
u,v	velocity components in the x-, y-directions, respectively
u_τ	friction velocity $(\tau_w/\rho_w)^{1/2}$
x,y	Cartesian coordinates
γ	parameter in the outer eddy viscosity formula, see (2.2.3b)
δ	boundary-layer thickness
δ^*	displacement thickness $\int_0^\infty [1 - (\rho u/\rho_e u_e)]dy$
$\varepsilon_m, \varepsilon_h$	eddy viscosity and eddy conductivity, respectively
$\varepsilon_m^+, \varepsilon_h^+$	dimensionless eddy viscosity and eddy conductivity, respectively
η	transformed y-coordinate
η_∞	transformed boundary-layer thickness
θ	momentum thickness, $\int_0^\infty (\rho u/\rho_e u_e) [1 - (u/u_e)]dy$
κ	Von Karman's constant
μ	dynamic viscosity
ν	kinematic viscosity
ρ	density

τ shear stress

ψ stream function

Subscripts

e outer edge of boundary layer

w wall

∞ free-stream conditions

Primes denote differentiation with respect to η

I. INTRODUCTION

Studies in separating boundary layers leading to accurate methods for predicting flow separation, predicting flows on the verge of separation, predicting separating and reattaching flows and predicting fully-separated flows find a large number of applications in aerodynamic problems and naturally in the design of both commercial and military airplanes. For example, an accurate determination of the separation point is very crucial in many problems since separation strongly influences the performance of aerodynamic configurations. A common procedure employed to determine this point is to solve the governing boundary-layer equations for a given external velocity distribution and find the point (if any) where the wall shear goes to zero. According to recent studies (for example, ref. 1), such a prediction can be done accurately for two-dimensional and axisymmetric laminar and turbulent flows. Such a capability should also exist shortly for three-dimensional flows in view of the considerable work being done in this area (see for example, refs. 2, 3, 4).

Predicting flows on the verge of separation (that is, flows with zero wall shear) is quite important in many problems. In the case of airfoils where it is desired to maximize the lift, it is necessary to compute the minimum distance over which a given pressure rise can be obtained without the flow separating. The most rapid pressure rise that it is possible to obtain occurs when the wall shear stress along the suction side of the airfoil decreases to zero. Therefore, it is of considerable interest to be able to calculate boundary layers with specified values for the wall shear that also decrease to zero. The Liebeck airfoils discussed in refs. 5 and 6 are designed on that principle.

Prediction of partially separating flows is one of the most difficult, yet rewarding tasks in aerodynamics. An economic and efficient operation of aerodynamic devices depends on smooth, streamlined flow. The upper limit of this efficient operating range is marked by the flow breakaway, called separation, or stall. The ideal attached-flow conditions are seldom attained in practice since a design is a series of compromises between conflicting requirements. As a specific example, consider an airplane wing whose maximum lift is determined by separation. If this wing was designed from the performance standpoint alone, simultaneous spanwise stall would be desirable. However, because nobody wants to stall without lateral control (by ailerons), wings are designed to have progressive stall from the wing root out. Operating with partially separated wings, especially for swept wings, creates in turn, longitudinal control problems because of shifts in the center of pressure. From the standpoint of design of control surfaces it is necessary to have methods for calculating overall forces on wings with partial separation. Today such a capability for calculating partially separating flows, even in two dimensions, does not exist.

Prediction of separating and reattaching flows is also very important. Two typical examples of such flows are a leading edge bubble and shock boundary-layer interaction. In both cases the boundary layer separates from the surface but reattaches after a short distance. Here it is important to know under what conditions the boundary layer reattaches or separates completely. The reattachment after a leading edge bubble on highly swept wings is relevant to the leading-edge vortex formation, and the absence of reattachment after a shock signifies shock-induced stall. Today a satisfactory prediction of separating and reattaching flows, even for two dimensions, again does not exist.

In recent years a number of studies on inverse boundary-layer flows have been conducted. Except for the integral-method approaches, most of these studies have been directed to laminar layers. In ref. 7, Catherall and Mangler solved the laminar boundary-layer equations in the usual way until the separation point was approached. By assuming that the displacement thickness behaves in a regular prescribed fashion in the region of the separation point, they calculated the pressure distribution in that region for the prescribed displacement thickness distribution. Their numerical solutions did not show any signs of a singular behavior at separation.

In ref. 8, Keller and Cebeci solved the laminar boundary-layer equations for a prescribed positive wall shear and in ref. 9, Klineberg and Steger solved them for a prescribed negative wall shear. In ref. 10, Carter presented numerical solutions of the laminar boundary-layer equations involving separation and reattachment. He obtained solutions with an inverse procedure in which he prescribed the displacement thickness or the wall shear. He compared his results with Klineberg and Steger's separated boundary-layer calculations[9] and with Briley's solution[11] of the Navier-Stokes equations for a separated region.

In ref. 12, Cebeci, Berkant, Silivri and Keller solved the turbulent boundary-layer equations for a prescribed positive wall shear. The only other turbulent boundary-layer calculations for flows with prescribed wall shear were made by Kuhn and Nielsen[13], by using an integral technique. However, unlike ref. 12, their solutions include negative wall shear as well as positive wall shear.

The work described in this report is one phase of studies on separating flows conducted under the contract N00014-74-A-0203-0001, NR215-234, from the Office of Naval Research. It deals with the calculation of laminar and turbulent boundary-layer flows for standard and inverse boundary-value problems, and is applicable to both incompressible and compressible flows. The standard boundary-layer problem considers the solution of the usual boundary-layer equations for a given external velocity distribution. The inverse problem considers the solution of the governing equations for assigned wall shear or for assigned displacement thickness. It provides a very useful and powerful method for calculating flows on the verge of separating.

The method, which is developed for two-dimensional flows, can easily be extended to axisymmetric flows. It also has the potential to be used in a number of problems that require inverse boundary-layer procedures. Some of them are:

1. Laminar flow control studies. Here the problem is to find the minimum suction rate to keep the flow laminar.
2. Design of ducts for a given pressure distribution.
3. Design of optimum ducts.
4. Calculation of attached duct-flows (inviscid and viscous) such as those in diffusers.
5. Design of two-dimensional and axisymmetric shapes.
6. Possible application to separated external flows.
7. Possible application to separated flows in ducts, i.e., diffusers.

II. GOVERNING EQUATIONS

2.1 Boundary-Layer Equations

The governing boundary-layer equations for steady, two-dimensional, compressible, laminar and turbulent boundary layers are the continuity, momentum and energy equations. They are given by:

Continuity

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\overline{\rho v}) = 0 \tag{2.1.1}$$

Momentum

$$\rho u \frac{\partial u}{\partial x} + \overline{\rho v} \frac{\partial u}{\partial y} = -\frac{dp}{dx} + \frac{\partial}{\partial y}\left[\mu \frac{\partial u}{\partial y} - \rho \overline{u'v'}\right] \tag{2.1.2}$$

Energy

$$\rho u \frac{\partial H}{\partial x} + \overline{\rho v} \frac{\partial H}{\partial y} = \frac{\partial}{\partial y}\left[\frac{\mu}{Pr}\frac{\partial H}{\partial y} - \rho \overline{v'H'} + \mu\left(1 - \frac{1}{Pr}\right) u \frac{\partial u}{\partial y}\right] \tag{2.1.3}$$

Here $-\rho\overline{u'v'}$ and $-\rho\overline{v'H'}$ denote the Reynolds stresses and

$$\overline{\rho v} = \rho v + \overline{\rho' v'}$$

The boundary conditions for (2.1.1) to (2.1.3) for zero mass transfer are:

$$y = 0 \qquad u, v = 0 \qquad H = H_w \quad \text{or} \quad (\partial H/\partial y)_w = \text{given} \tag{2.1.4a}$$

$$y \to \infty \qquad u \to u_e(x) \qquad H \to H_e \tag{2.1.4b}$$

2.2 Closure Assumptions for the Reynolds Stresses

The solution of the system given by (2.1.1) to (2.1.4) requiresclosure assumptions for the Reynolds stresses, $-\rho\overline{u'v'}$ and $-\rho\overline{v'H'}$. In our study we use eddy viscosity (ε_m) and eddv conductivity (ε_H) concepts and define

$$-\rho\overline{u'v'} = \rho\varepsilon_m \frac{\partial u}{\partial y}, \qquad -\rho\overline{v'H'} = \rho\varepsilon_H \frac{\partial H}{\partial y} \tag{2.2.1}$$

and relate ε_m and ε_H to a turbulent Prandtl number Pr_t by

$$Pr_t = \varepsilon_m/\varepsilon_H \tag{2.2.2}$$

According to the eddy-viscosity formulation of Cebeci and Smith[14], the turbulent boundary layer is divided into two regions, called inner and outer regions, and the eddy viscosity is defined by separate formulas in each region. They are:

$$\varepsilon_m = \begin{cases} (\varepsilon_m)_i = L^2 \left|\dfrac{\partial u}{\partial y}\right| , & (\varepsilon_m)_i \leq (\varepsilon_m)_o \qquad (2.2.3a) \\ (\varepsilon_m)_o = \alpha \left| \displaystyle\int_0^\infty (u_e - u)dy \right| , & (\varepsilon_m)_i > (\varepsilon_m)_o \qquad (2.2.3b) \end{cases}$$

Here

$$\begin{aligned} L &= \kappa y \, [1 - \exp(-y/A)] \\ A &= A^+ \frac{\nu}{N} \left(\frac{\tau_w}{\rho_w}\right)^{-1/2} \left(\frac{\rho}{\rho_w}\right)^{1/2} \\ N &= [1 - 11.8(\mu_w/\mu_e)(\rho_e/\rho_w)^2 \, p^+]^{1/2} \\ p^+ &= \frac{\nu_e u_e}{u_\tau^3} \frac{du_e}{dx} , \qquad u_\tau = \left(\frac{\tau_w}{\rho_w}\right)^{1/2} \end{aligned} \tag{2.2.4}$$

In (2.2.3b) and (2.2.4) α, κ and A^+ are "universal" constants equal to 0.0168, 0.40, and 26, respectively, for high Reynolds number flows, $R_\theta > 5000$. To compute flows at low Reynolds numbers, one can modify these three coefficients as discussed in ref. 3, p. 221.

The eddy-viscosity formulas (2.2.3) can also be modified to compute transitional boundary layers as well as boundary layers in which the streamwise wall curvature becomes important. Again for a detailed discussion see ref. 3, p. 232.

2.3 Transformation of the Governing Equations

Before we solve the system given by (2.1.1) through (2.1.4) with the Reynolds stresses replaced by (2.2.1), we introduce the Falkner-Skan transformation to remove the singularity at $x = 0$ and to stretch the coordinate in the y direction; we define:

$$x = x \qquad d\eta = \left(\frac{u_e}{\rho_e \mu_e x}\right)^{1/2} \rho dy \tag{2.3.1}$$

This transformation not only allows the calculations to be started very easily at the leading edge or at the stagnation point but also removes the large variation in boundary-layer thickness along the surface. In transformed variables, the velocity profiles and temperature profiles do not change "much" as the calculations proceed in the x-direction. This results in small computation times, and allows larger spacings to be taken in the x-direction.

We next define the stream function ψ by

$$\rho u = \frac{\partial \psi}{\partial y}, \qquad \overline{\rho v} = -\frac{\partial \psi}{\partial x} \tag{2.3.2}$$

and a dimensionless stream function $f(x,\eta)$ by

$$\psi = (\rho_e \mu_e u_e x)^{1/2} f(x,\eta) \tag{2.3.3}$$

With the relations (2.3.1) through (2.3.3), we can write the momentum and energy equations as

Momentum

$$(bf'')' + P_1 ff'' + P_2[c - (f')^2] = x\left(f' \frac{\partial f'}{\partial x} - f'' \frac{\partial f}{\partial x}\right) \tag{2.3.4}$$

Energy

$$(eg' + df'f'')' + P_1 fg' = x\left(f' \frac{\partial g}{\partial x} - g' \frac{\partial f}{\partial x}\right) \tag{2.3.5}$$

Here primes denote differentiation with respect to η and

$$f' = \frac{u}{u_e}, \qquad g \equiv \frac{H}{H_e}, \qquad c = \frac{\rho_e}{\rho}, \qquad C \equiv \frac{\rho\mu}{\rho_e \mu_e} \tag{2.3.6a}$$

$$b = (1 + \varepsilon_m^+)C, \qquad e = \frac{C}{Pr}\left(1 + \varepsilon_m^+ \frac{Pr}{Pr_t}\right), \qquad d = \frac{u_e^2}{H_e}\left(1 - \frac{1}{Pr}\right) \tag{2.3.6b}$$

$$P_1 \equiv \frac{1}{2}\left[1 + P_2 + \frac{x}{\rho_e\mu_e}\frac{d}{dx}(\rho_e\mu_e)\right] \qquad P_2 = \frac{x}{u_e}\frac{du_e}{dx} \tag{2.3.6c}$$

Similarly the boundary conditions in (2.1.4) become

Momentum

$$\eta = 0 \qquad f = 0 \qquad f' = 0 \tag{2.3.7a}$$

$$\eta = \infty \qquad f' = 1 \tag{2.3.7b}$$

Energy

$$\eta = 0 \qquad g = g_w \qquad \text{or} \qquad g_w' = \text{given} \tag{2.3.8a}$$

$$\eta = \eta_\infty \qquad g = 1 \tag{2.3.8b}$$

In terms of transformed variables, the inner and outer eddy viscosity formulas in (2.2.3) can be written in dimensionless form as

$$\varepsilon_m^+ \equiv \begin{cases} (\varepsilon_m^+)_i = \kappa^2 C \left(\frac{\mu_e}{\mu}\right)^2 R_x^{1/2} \frac{1}{C^2} |f''| I_1^2 [1 - \exp(-y/A)]^2 & (2.3.9a) \\ (\varepsilon_m^+)_o = \alpha C \left(\frac{\mu_e}{\mu}\right)^2 R_x^{1/2} \left| \int_0^\infty \frac{\rho_e}{\rho}(1 - f')d\eta \right| & (2.3.9b) \end{cases}$$

where $\varepsilon_m^+ = \varepsilon_m/\nu$ and

$$\frac{y}{A} = C^{1/2}\left(\frac{\mu_w}{\mu}\right)^{1/2}\left(\frac{\mu_e}{\mu}\right)\left(\frac{\rho_w}{\rho_e}\right)^{1/2} I_1 R_x^{1/4}(f_w'')^{1/2} N/A^+$$

$$N = \left[1 - 11.8\,\frac{\mu_w}{\mu_e}\left(\frac{\rho_e}{\rho_w}\right)^2 p^+\right]^{1/2}, \qquad p^+ = \frac{\nu_e}{u_\tau^3}\frac{u_e^2}{x}P_2 \tag{2.3.10}$$

$$u_\tau = u_e\left(\frac{\mu_w}{\mu_e}\frac{f_w''}{R_x^{1/2}}\right)^{1/2}, \qquad I_1 = \int_0^\eta \frac{\rho_e}{\rho}d\eta, \qquad R_x = \frac{u_e x}{\nu_e}$$

III. STANDARD AND INVERSE PROBLEMS

The system given by (2.3.4), (2.3.5), (2.3.7) and (2.3.8) with specified $u_e(x)$ or $P_2(x)$ is the typical two-dimensional boundary-layer problem for laminar and turbulent flows. For convenience we shall call it the <u>standard problem</u>.

There are a number of problems that require inverse procedures in viscous flows. One type of <u>inverse problem</u> results from requiring that the local skin-friction coefficient c_f defined by

$$c_f = \frac{\tau_w}{(1/2)\rho_e u_e^2} \tag{3.1.1}$$

be specified. In terms of transformed variables (3.1.1) becomes

$$c_f = \frac{2f''_w C_w}{\sqrt{R_x}} \tag{3.1.2}$$

Another type of inverse problem results from requiring that the displacement thickness defined by

$$\delta^* = \int_0^\infty \left(1 - \frac{\rho u}{\rho_e u_e}\right) dy \tag{3.1.3}$$

be specified. In terms of transformed variables (3.1.3) becomes

$$\delta^* = \frac{x}{\sqrt{R_x}} \int_0^{\eta_\infty} \left(\frac{\rho_e}{\rho} - f'\right) d\eta \tag{3.1.4}$$

Other inverse problems can be formulated for the problems discussed at the end of Section I. The system given by (2.3.4), (2.3.5), (2.3.7), (2.3.8) and (3.1.2) or (3.1.4) is overdetermined and we cannot specify $P_2(x)$ (i.e., $u_e(x)$) arbitrarily. Rather we must determine $P_2(x)$ as well as $f(x,\eta)$ to solve the system. In our study we use Newton's method and determine the unknowns by the procedure discussed in the next section.

3.1 Newton's Method for the Inverse Problem

To describe our numerical approach to the problem of the specified c_f case, let us assume that at $x = x_{n-1}$ we are given the profiles of $\bar{f}$, f', f'', g, g', the pressure gradient $P_2(x_{n-1})$ and the velocity $u_e(x_{n-1})$. At $x = x_n$ we seek an accurate approximation to the solution of (2.3.4), (2.3.5) subject to (2.3.7), (2.3.8) for a given $c_f(x)$. To start the calculations, it is necessary to know $P_2(x)$ and $u_e(x)$. The latter is necessary since R_x is a function of u_e. In our method we assume $P_2(x)$ and calculate $u_e(x)$ from the definition of $P_2(x)$ in (2.3.6c). Using central differences we approximate P_2 and solve it for $u_e^n = u_e(x_n)$ to get

$$u_e^n = -u_e^{n-1} \frac{P_2^{n-1/2} + 2\alpha_n}{P_2^{n-1/2} - 2\alpha_n} \tag{3.1.5}$$

where

$$\alpha_n = \frac{x_{n-1/2}}{x_n - x_{n-1}}, \quad P_2^{n-1/2} = \frac{1}{2}(P_2^n + P_2^{n-1}), \quad x_{n-1/2} = \frac{1}{2}(x_n + x_{n-1}) \tag{3.1.6}$$

Once $P_2(x)$ and $u_e(x)$ are known, then the standard problem (2.3.4), (2.3.5) subject to (2.3.7), (2.3.8) can be solved. The numerical method used to do this will be described in Section IV. Let us denote the solution of the standard problem by

$$f(x,\eta) = \mathcal{B}[x,\eta,P_2(x)] \tag{3.1.7}$$

Using this solution, we can now calculate c_f (which we shall denote by c_{f_c}) from (3.1.2). Recalling that the desired value for the skin-friction coefficient is $c_f(x)$, we form:

$$\phi[P_2(x)] \equiv c_{f_c} - c_f \tag{3.1.8}$$

and seek $P_2(x)$ such that $\phi[P_2(x)] = 0$ on $x > 0$.

To solve $\phi[P_2(x)] = 0$, we use Newton's method. With some estimate $P_2^{(0)}(x)$ of the desired pressure gradient, we define the sequence $P_2^{(\nu)}(x)$ by setting

$$P_2^{\nu+1}(x) = P_2^{\nu}(x) - \frac{\phi[P_2^{(\nu)}(x)]}{(\partial/\partial P_2)\{\phi[P_2^{(\nu)}(x)]\}} \tag{3.1.9}$$

The derivative of ϕ with respect to P_2 can be obtained from (3.1.8) by making use of the relation given by (3.1.2) and (3.1.5). This gives

$$\frac{\partial\phi}{\partial P_2} = \frac{2C_w}{\sqrt{R_x}}\left\{-F''_w + f''_w \frac{u_e^{n-1}}{u_e^n} - \left[\frac{2\alpha_n}{(P_2^{n-1/2} - 2\alpha_n)^2}\right]\right\} \tag{3.1.10}$$

where

$$F''_w = \frac{\partial f''_w}{\partial P_2} \tag{3.1.11}$$

To summarize one step of iteration of Newton's method, we first estimate a value for $P_2(x_n)$, then calculate u_e^n from (3.1.5), and obtain a solution of (2.3.4), (2.3.5) subject to (2.3.7) and (2.3.8). The solution yields a $c_{f_c}^{(\nu)}$ according to (3.1.2). From this result and from the desired value $c_f(x_n)$, we find ϕ from (3.1.8). It is then clear that the next value of $P_2(x_n)$ can be calculated from (3.1.9), provided that $\partial\phi/\partial P_2$ is known. In Section 4.3 we shall discuss its calculation. The iteration process is repeated until

$$|P_2^{(\nu+1)}(x_n) - P_2^{(\nu)}(x_n)| < \gamma_1 \tag{3.1.12}$$

where γ_1 is a small error tolerance.

Our procedure for the specified δ^* case, is similar to the procedure for the specified c_f case. The difference in the procedure starts after we get the solution (3.1.7). Denoting the calculated value of δ^* by δ^*_c, and the desired value by δ^*, we form

$$\phi[P_2(x)] \equiv \delta^*_c - \delta^* \tag{3.1.13}$$

We obtain $P_2^{\nu+1}(x)$ from the expression given by (3.1.9). To find the derivative of ϕ with respect to P_2 from (3.1.13), we first write (3.1.4) as

$$\delta^* = \frac{xA}{\sqrt{R_x^*}} \tag{3.1.14}$$

where

$$A \equiv \int_0^\infty (c - u)d\eta \tag{3.1.15}$$

Differentiating (3.1.14) with respect to P_2 and using (3.1.13), we get

$$\frac{\partial\phi}{\partial P_2} = \frac{x}{\sqrt{R_x}}\left[\frac{\partial A}{\partial P_2} - \frac{A}{2}\frac{1}{u_e^n}\frac{\partial u_e}{\partial P_2}\right] \tag{3.1.16}$$

From (3.1.5) and (3.1.15) it follows that

$$\frac{\partial u_e}{\partial P_2} = -\,u_e^{n-1}\,\frac{4\alpha_n}{(P_2^{n+(1/2)} - 2\alpha_n)^2} \tag{3.1.17}$$

$$\frac{\partial A}{\partial P_2} = -F(\eta_\infty) \tag{3.1.18}$$

where

$$F \equiv \frac{\partial f}{\partial P_2}$$

IV. SOLUTION OF THE GOVERNING EQUATIONS FOR THE STANDARD PROBLEM

4.1 Numerical Formulation

We use a very efficient and accurate numerical method to solve the governing equations. This is a two-point finite difference method developed by H. B. Keller[15] and applied to the boundary-layer equations by Keller and Cebeci (see, for example, references 16, 17).

According to this method we introduce new independent variables $u(x,\eta)$, $v(x,\eta)$, $t(x,\eta)$ so that (2.3.4) and (2.3.5) can be written as a first-order system

$$f' = u \tag{4.1.1a}$$

$$u' = v \tag{4.1.1b}$$

$$g' = t \tag{4.1.1c}$$

$$(bv)' + P_1 fv + P_2 (c - u^2) = x\left(u \frac{\partial u}{\partial x} - v \frac{\partial f}{\partial x}\right) \tag{4.1.1d}$$

$$(et + duv)' + P_1 ft = x\left(u \frac{\partial g}{\partial x} - t \frac{\partial f}{\partial x}\right) \tag{4.1.1e}$$

On the net rectangle shown in Figure 1, we denote the net points by

$$\begin{aligned} &x_0 = 0, \qquad x_n = x_{n-1} + k_n, \qquad n = 1, 2, \ldots, N. \\ &\eta_0 = 0, \qquad \eta_j = \eta_{j-1} + h_j, \qquad j = 1, 2, \ldots, J: \eta_J = \eta_\infty \end{aligned} \tag{4.1.2}$$

Here the net spacings k_n and h_j are completely arbitrary. The quantities (f, u, v, g, t) at points (x_n, η_j) of the net are approximated by net functions denoted by $(f_j^n, u_j^n, v_j^n, g_j^n, t_j^n)$. We also employ the notation for points and quantities midway between net points and for <u>any</u> net function q_j^n:

$$\begin{aligned} &x_{n-\frac{1}{2}} \equiv \frac{1}{2}(x_n + x_{n-1}), \qquad \eta_{j-\frac{1}{2}} \equiv \frac{1}{2}(\eta_j + \eta_{j-1}) \\ &q_j^{n-\frac{1}{2}} \equiv \frac{1}{2}(q_j^n + q_j^{n-1}), \qquad q_{j-\frac{1}{2}}^n \equiv \frac{1}{2}(q_j^n + q_{j-1}^n) \end{aligned} \tag{4.1.3}$$

$$\alpha_n \equiv \frac{x_{n-\frac{1}{2}}}{x_n - x_{n-1}} = \frac{x_{n-\frac{1}{2}}}{k_n}$$

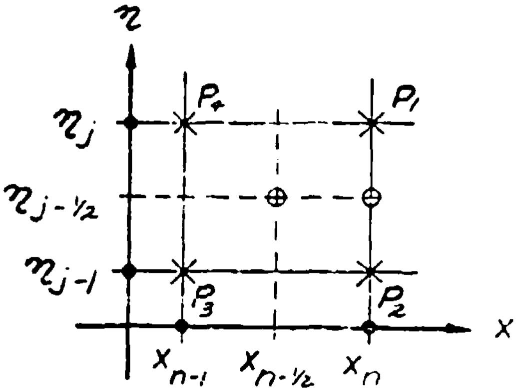


Fig. 1. Net rectangle for difference approximations

The difference equations that are to approximate (4.1.1) are formulated by considering one mesh rectangle as in Fig. 1. We approximate (4.1.1a,b,c) using centered difference quotients and average about the midpoint $(x_n, \eta_{j-\frac{1}{2}})$ of the segment P_1P_2 as follows:

$$h_j^{-1}\,(f_j^n - f_{j-1}^n) = u_{j-\frac{1}{2}}^n \tag{4.1.4a}$$

$$h_j^{-1}\,(u_j^n - u_{j-1}^n) = v_{j-\frac{1}{2}}^n \tag{4.1.4b}$$

$$h_j^{-1}\,(g_j^n - g_{j-1}^n) = t_{j-\frac{1}{2}}^n \tag{4.1.4c}$$

Similarly (4.1.1d,e) are approximated by centering about the midpoint $(x_{n-\frac{1}{2}}, \eta_{j-\frac{1}{2}})$ of the rectangle $P_1P_2P_3P_4$

$$h_j^{-1}\,(b_j^n v_j^n - b_{j-1}^n v_{j-1}^n) + (P_1^n + \alpha_n)(fv)_{j-\frac{1}{2}}^n - (P_2^n + \alpha_n)(u^2)_{j-\frac{1}{2}}^n$$

$$+\ \alpha_n(v_{j-\frac{1}{2}}^{n-1} f_{j-\frac{1}{2}}^n - f_{j-\frac{1}{2}}^{n-1} v_{j-\frac{1}{2}}^n) = R_{j-\frac{1}{2}}^{n-1} - P_2^n\, c_{j-\frac{1}{2}}^n \tag{4.1.4d}$$

$$h_j^{-1}(e_j^n t_j^n - e_{j-1}^n t_{j-1}^n) + (P_1^n + \alpha_n)(ft)_{j-\frac{1}{2}}^n - \alpha_n\Big[(ug)_{j-\frac{1}{2}}^n + u_{j-\frac{1}{2}}^{n-1} g_{j-\frac{1}{2}}^n$$

$$- g_{j-\frac{1}{2}}^{n-1} u_{j-\frac{1}{2}}^n + f_{j-\frac{1}{2}}^{n-1} t_{j-\frac{1}{2}}^n - t_{j-\frac{1}{2}}^{n-1} f_{j-\frac{1}{2}}^n\Big] = T_{j-\frac{1}{2}}^{n-1} \qquad (4.1.4e)$$

where

$$R_{j-\frac{1}{2}}^{n-1} = \alpha_n\Big[(fv)_{j-\frac{1}{2}}^{n-1} - (u^2)_{j-\frac{1}{2}}^{n-1}\Big] - \Big\{h_j^{-1}(b_j^{n-1} v_j^{n-1} - b_{j-1}^{n-1} v_{j-1}^{n-1}) + P_1^{n-1}(fv)_{j-\frac{1}{2}}^{n-1}$$

$$+ P_2^{n-1}\Big[c_{j-\frac{1}{2}}^{n-1} - (u^2)_{j-\frac{1}{2}}^{n-1}\Big]\Big\} \qquad (4.1.5a)$$

$$T_{j-\frac{1}{2}}^{n-1} = \alpha_n\Big[(ft)_{j-\frac{1}{2}}^{n-1} - (ug)_{j-\frac{1}{2}}^{n-1}\Big] - 2\Big[(duv)'\Big]_{j-\frac{1}{2}}^{n-\frac{1}{2}} - \Big[h_j^{-1}(e_j^{n-1} t_j^{n-1} - e_{j-1}^{n-1} t_{j-1}^{n-1})$$

$$+ P_1^{n-1}(ft)_{j-\frac{1}{2}}^{n-1}\Big] \qquad (4.1.5b)$$

Equations (4.1.4) are imposed for $j = 1,2,\ldots,J$. The boundary conditions (2.3.7) and (2.3.8) yield, at $x = x_n$,

$$f_o^n = 0, \quad u_o^n = 0, \quad u_J^n = 1, \quad g_o^n = g_w^n \text{ or } t_o^n = t_w^n, \quad g_J^n = 1. \qquad (4.1.6)$$

4.2 Solution of the Difference Equations

If we assume $(f_j^{n-1}, u_j^{n-1}, v_j^{n-1}, g_j^{n-1}, t_j^{n-1})$ to be known for $0 \le j \le J$, then (4.1.4) for $1 \le j \le J$ and the boundary conditions (4.1.6) yield a nonlinear algebraic system of $5J+5$ equations in as many unknowns $(f_j^n, u_j^n, v_j^n, g_j^n, t_j^n)$. The system can be solved very effectively by using Newton's method. We introduce iterates $[f_j^{(i)}, u_j^{(i)}, v_j^{(i)}, g_j^{(i)}, t_j^{(i)}]$, $i = 0,1,2,\ldots$, with initial values for the specified wall temperature g_w, say

$$f_0^{(0)} = 0, \quad u_0^{(0)} = 0, \quad v_0^{(0)} = v_0^{n-1}, \quad g_0^{(0)} = g_w, \quad t_0^{(0)} = t_0^{n-1},$$

$$f_j^{(0)} \equiv f_j^{n-1}, \quad u_j^{(0)} \equiv u_j^{n-1}, \quad v_j^{(0)} \equiv v_j^{n-1}, \quad g_j^{(0)} = g_j^{n-1}$$

$$t_j^{(0)} = t_j^{n-1}, \quad 1 \leq j \leq J-1$$

$$f_J^{(0)} = f_J^{n-1}, \quad u_J^{(0)} = 0, \quad v_J^{(0)} = v_J^{n-1}, \quad g_J^{(0)} = 0, \quad t_J^{(0)} = t_J^{n-1} \tag{4.2.1}$$

For the higher order iterates we set

$$f_j^{(i+1)} = f_j^{(i)} + \delta f_j^{(i)}, \quad u_j^{(i+1)} = u_j^{(i)} + \delta u_j^{(i)}, \quad v_j^{(i+1)} = v_j^{(i)} + \delta v_j^{(i)}$$

$$g_j^{(i+1)} = g_j^{(i)} + \delta g_j^{(i)}, \quad t_j^{(i+1)} = t_j^{(i)} + \delta t_j^{(i)}$$

Then we insert these expressions in place of $[f_j, u_j, v_j, g_j, t_j]$ in (4.1.4) and drop the terms that are quadratic in $[\delta f_j^{(i)}, \delta u_j^{(i)}, \delta v_j^{(i)}, \delta g_j^{(i)}, \delta t_j^{(i)}]$. This procedure yields the following linear system:

$$\delta f_j - \delta f_{j-1} - \frac{h_j}{2}(\delta u_j + \delta u_{j-1}) = (r_1)_{j-\frac{1}{2}} \tag{4.2.2a}$$

$$\delta u_j - \delta u_{j-1} - \frac{h_j}{2}(\delta v_j + \delta v_{j-1}) = (r_2)_{j-\frac{1}{2}} \tag{4.2.2b}$$

$$\delta g_j - \delta g_{j-1} - \frac{h_j}{2}(\delta t_j + \delta t_{j-1}) = (r_3)_{j-\frac{1}{2}} \tag{4.2.2c}$$

$$(\zeta_1)_j \, \delta v_j + (\zeta_2)_j \, \delta v_{j-1} + (\zeta_3)_j \, \delta f_j + (\zeta_4)_j \, \delta f_{j-1} + (\zeta_5)_j \, \delta u_j + (\zeta_6)_j \, \delta u_{j-1} = (r_4)_{j-\frac{1}{2}} \tag{4.2.2d}$$

$$(\beta_1)_j \, \delta t_j + (\beta_2)_j \, \delta t_{j-1} + (\beta_3)_j \, \delta f_j + (\beta_4)_j \, \delta f_{j-1} + (\beta_5)_j \, \delta g_j + (\beta_6)_j \, \delta g_{j-1} + (\beta_7)_j \, \delta u_j + (\beta_8)_j \, \delta u_{j-1} = (r_5)_{j-\frac{1}{2}} \tag{4.2.2e}$$

for $j = 1,2,\ldots,J-1$. Here for convenience we have dropped the superscripts i and n. The coefficients $(\zeta_k)_j$ $(k = 1$ to $6)$ of the differenced momentum equation are:

$$(\zeta_1)_j = b_j + \frac{h_j}{2}\left[(P_1 + \alpha_n)f_j - \alpha_n f^{n-1}_{j-\frac{1}{2}}\right]$$

$$(\zeta_2)_j = -b_{j-1} + \frac{h_j}{2}\left[(P_1 + \alpha_n)f_{j-1} - \alpha_n f^{n-1}_{j-\frac{1}{2}}\right]$$

$$(\zeta_3)_j = \frac{h_j}{2}\left[(P_1 + \alpha_n)v_j + \alpha_n v^{n-1}_{j-\frac{1}{2}}\right]$$

$$(\zeta_4)_j = \frac{h_j}{2}\left[(P_1 + \alpha_n)v_{j-1} + \alpha_n v^{n-1}_{j-\frac{1}{2}}\right] \tag{4.2.3}$$

$$(\zeta_5)_j = -h_j(P_2 + \alpha_n)u_j$$

$$(\zeta_6)_j = -h_j(P_2 + \alpha_n)u_{j-1}$$

The coefficients $(\beta_k)_j$ $(k = 1$ to $8)$ of the differenced energy equation are

$$(\beta_1)_j = e_j + \frac{h_j}{2}\left[(P_1 + \alpha_n)f_j - \alpha_n f^{n-1}_{j-\frac{1}{2}}\right]$$

$$(\beta_2)_j = -e_{j-1} + \frac{h_j}{2}\left[(P_1 + \alpha_n)f_{j-1} - \alpha_n f^{n-1}_{j-\frac{1}{2}}\right]$$

$$(\beta_3)_j = \frac{h_j}{2}\left[(P_1 + \alpha_n)t_j + \alpha_n t^{n-1}_{j-\frac{1}{2}}\right]$$

$$(\beta_4)_j = \frac{h_j}{2}\left[(P_1 + \alpha_n)t_{j-1} + \alpha_n t^{n-1}_{j-\frac{1}{2}}\right] \tag{4.2.4}$$

$$(\beta_5)_j = -\frac{h_j}{2}\alpha_n\left(u_j + u^{n-1}_{j-\frac{1}{2}}\right)$$

$$(\beta_6)_j = -\frac{h_j}{2}\alpha_n\left(u_{j-1} + u^{n-1}_{j-\frac{1}{2}}\right)$$

$$(\beta_7)_j = -\frac{h_j}{2}\alpha_n\left(g_j - g^{n-1}_{j-\frac{1}{2}}\right)$$

$$(\beta_8)_j = -\frac{h_j}{2}\alpha_n\left(g_{j-1} - g^{n-1}_{j-\frac{1}{2}}\right) \tag{4.2.4}$$

The coefficients $(r_k)_j$ $(k = 1 \text{ to } 5)$ are:

$$(r_1)_j = h_j u_{j-\frac{1}{2}} - f_j + f_{j-1}$$

$$(r_2)_j = h_j v_{j-\frac{1}{2}} - u_j + u_{j-1}$$

$$(r_3)_j = h_j t_{j-\frac{1}{2}} - g_j + g_{j-1} \tag{4.2.5}$$

$$(r_4)_j = h_j\left(R^{n-1}_{j-\frac{1}{2}} - P_2 c_{j-\frac{1}{2}}\right) - \Big\{ b_j v_j - b_{j-1}v_{j-1} + h_j\Big[(P_1 + \alpha_n)(fv)_{j-\frac{1}{2}}$$
$$- (P_2 + \alpha_n)(u^2)_{j-\frac{1}{2}} - \alpha_n\left(-v^{n-1}_{j-1} f^n_{j-\frac{1}{2}} + f^{n-1}_{j-i} v^n_{j-\frac{1}{2}}\right)\Big]\Big\}$$

$$(r_5)_j = h_j T^{n-1}_{j-\frac{1}{2}} - \Big[e_j t_j - c_{j-1}t_{j-1} + h_j\ \Big\{(P_1 + \alpha_n)(ft)_{j-\frac{1}{2}}$$
$$- \alpha_n\left[(ug)_{j-\frac{1}{2}} - u_{j-\frac{1}{2}}g^{n-1}_{j-\frac{1}{2}} + g_{j-\frac{1}{2}}u^{n-1}_{j-\frac{1}{2}} + f^{n-1}_{j-\frac{1}{2}}t_{j-\frac{1}{2}} - t^{n-1}_{j-\frac{1}{2}}f_{j-\frac{1}{2}}\right]\Big\}\Big]$$

The boundary conditions for the specified wall temperature become

$$\delta f_0 = 0, \quad \delta g_0 = 0, \quad \delta u_0 = 0, \quad \delta u_J = 0, \quad \delta g_J = 0 \tag{4.2.6}$$

The boundary conditions for the specified heat flux become

$$\delta f_0 = 0, \quad \delta t_0 = 0, \quad \delta u_0 = 0, \quad \delta u_J = 0, \quad \delta g_J = 0 \tag{4.2.7}$$

We use the block elimination method discussed by Isaacson and Keller[18] to solve the linear system (4.2.2), and (4.2.6) or (4.2.7) depending on the boundary conditions. For completeness we present the block elimination method for our problem in Appendix A.

V. SOLUTION OF THE GOVERNING EQUATIONS FOR THE INVERSE PROBLEM

5.1 Variational Equations

In order to calculate $\partial\phi/\partial P_2$ in (3.1.10) or in (3.1.16) it is necessary to know F''_w for the specified c_f case or $F(\eta_e)[\partial f/\partial P_2(\eta_\infty)]$ for the specified δ^* case. For this reason we take the derivative of (4.1.4) with respect to P_2^n. This leads to the following linear difference equations, known as the variational equations for (4.1.4):

$$h_j^{-1}\left(F_j^n - F_{j-1}^n\right) = U_{j-\frac{1}{2}}^n \tag{5.1.1a}$$

$$h_j^{-1}\left(U_j^n - U_{j-1}^n\right) = V_{j-\frac{1}{2}}^n \tag{5.1.1b}$$

$$h_j^{-1}\left(G_j^n - G_{j-1}^n\right) = T_{j-\frac{1}{2}}^n \tag{5.1.1c}$$

$$\begin{aligned} h_j^{-1}&\left(b_j^n V_j^n - b_{j-1}^n V_{j-1}^n\right) + \frac{1}{2}\left(P_1^n + \alpha_n\right)\left[f_j^n V_j^n + v_j^n F_j^n + f_{j-1}^n V_{j-1}^n + v_{j-1}^n F_{j-1}^n\right] \\ &+ \frac{1}{2}(fv)_{j-\frac{1}{2}}^n - (u^2)_{j-\frac{1}{2}}^n - \left(P_2^n + \alpha_n\right)\left(u_j^n U_j^n + u_{j-1} U_{j-1}^n\right) \\ &+ \frac{\alpha_n}{2}\left[v_{j-\frac{1}{2}}^{n-1}\left(F_j^n + F_{j-1}^n\right) - f^{n-1}\left(V_j^n + V_{j-1}^n\right)\right] = -c_{j-\frac{1}{2}}^n \end{aligned} \tag{5.1.1d}$$

$$\begin{aligned} h_j^{-1}&\left(e_j^n T_j^n - e_{j-1}^n T_{j-1}^n\right) + \frac{1}{2}\left(P_1^n + \alpha_n\right)\left[f_j^n T_j^n + t_j^n F_j^n + f_{j-1}^n T_{j-1}^n + t_{j-1}^n F_{j-1}^n\right] \\ &+ \frac{1}{2}(ft)_{j-\frac{1}{2}}^n - \frac{\alpha_n}{2}\left[u_j^n G_j^n + g_j^n U_j^n + u_{j-1}^n G_{j-1}^n + g_{j-1}^n U_{j-1}^n + u_{j-\frac{1}{2}}^{n-1}\left(G_j^n + G_{j-1}^n\right)\right. \\ &\left. - g_{j-\frac{1}{2}}^{n-1}\left(U_j^n + U_{j-1}^n\right) + f_{j-\frac{1}{2}}^{n-1}\left(T_j^n + T_{j-1}^n\right) - t_{j-\frac{1}{2}}^{n-1}\left(F_j^n + F_{j-1}^n\right)\right] = 0 \end{aligned} \tag{5.1.1e}$$

Equations (5.1.1a to c) can be written as

$$F_j^n - F_{j-1}^n - \frac{h_j}{2}\left(U_j^n + U_{j-1}^n\right) = 0 \tag{5.1.2a}$$

$$U_j^n - U_{j-1}^n - \frac{h_j}{2}\left(V_j^n + V_{j-1}^n\right) = 0 \tag{5.1.2b}$$

$$G_j^n - G_{j-1}^n - \frac{h_j}{2}\left(T_j^n + T_{j-1}^n\right) = 0 \tag{5.1.2c}$$

Equations (5.1.1d, e) can be rearranged in a form similar to (4.2.2d, e), that is:

$$(\zeta_1)_j V_j^n + (\zeta_2)_j V_{j-1}^n + (\zeta_3)_j F_j^n + (\zeta_4)_j F_{j-1}^n + (\zeta_5)_j U_j^n + (\zeta_6)_j U_{j-1}^n = (r_4)_{j-\frac{1}{2}} \tag{5.1.2d}$$

$$(\beta_1)_j T_j^n + (\beta_2) T_{j-1}^n + (\beta_3)_j F_j^n + (\beta_4)_j F_{j-1}^n + (\beta_5)_j G_j^n + (\beta_6)_j G_{j-1}^n + (\beta_7)_j U_j^n + (\beta_8)_j U_{j-1}^n = 0 \tag{5.1.2e}$$

Here

$$F \equiv \frac{\partial f}{\partial P_2}, \qquad U = \frac{\partial U}{\partial P_2}, \qquad V = \frac{\partial v}{\partial P_2}$$

and the coefficients ζ_k and β_k are the same as those given by (4.2.3) and (4.2.4). The coefficient $(r_4)_{j-\frac{1}{2}}$ is defined by

$$(r_4)_{j-\frac{1}{2}} = -c_{j-\frac{1}{2}}^n - \frac{1}{2}(fv)_{j-\frac{1}{2}}^n + (u^2)_{j-\frac{1}{2}}^n \tag{5.1.3}$$

Similarly the boundary conditions (4.1.6) become

$$F_o^n = 0, \quad U_o^n = 0, \quad U_J^n = 0, \quad G_o^n = 0 \quad \text{or} \quad T_o^n = 0, \quad G_J^n = 0 \tag{5.1.4}$$

As pointed out in Ref. 12, an alternative set of variational equations can be obtained by first taking the derivative of (4.1.1) and then differencing the resulting equations. However, this procedure does not necessarily yield a good approximation to the desired derivative, $V = \partial v/\partial P_2$ or $F = \partial f/\partial P_2$. In the limit, as h_j and $k_n \to 0$, both procedures yield the same result. But as is reported in Ref. 12, for the actual numerical calculations, the present procedure gives precisely the derivatives required for Newton's method while the other procedure may not. This is, in fact, one of the basic differences between what is sometimes called 'quasi-linearization" and our exact application of Newton's method. Thus the "quasi-linearized" iterations may not converge quadratically (as was found to be the case in references 12 and 17, but our present iterates do show superlinear convergence.

The system (5.1.2), (5.1.4) again forms a block tridiagonal system (with 5 x 5 blocks) that is easily solved by the block elimination method described in Appendix A.

VI. RESULTS FOR STANDARD AND INVERSE PROBLEMS

6.1 Grid Across the Boundary Layer

While the numerical scheme employed here is a general one in that any type of grid can be used in the η-direction (also in the x-direction), we have chosen a grid previously used by the author and his associates[14]. This grid has the property that the ratio of lengths of any two adjacent intervals is a constant, that is,

$$h_j = Kh_{j-1} \tag{6.1.1}$$

The distance to the j-th η-line is given by the following formula:

$$\eta_j = h_1 \frac{K^j - 1}{K - 1} \qquad j = 1, 2, 3, \ldots, J \qquad K > 1 \tag{6.1.2}$$

There are two parameters: h_1, the length of the first $\Delta\eta$-step, and K, the ratio of two successive steps. The total number of points J are calculated by the following formula:

$$J = \frac{\ln[1 + (K - 1)\eta_\infty/h_1]}{\ln K} \tag{6.1.3}$$

In our calculations we select the parameters h_1 and K and calculate the η_∞. Several runs with a different number of points across the boundary layer showed that (see ref. 14) approximately 30 to 40 points are sufficient for turbulent flows. A typical value of h_1 is 0.01, provided that the Reynolds number is not very large, say less than 10^7. At higher Reynolds numbers, it is better to use a smaller value of h_1, say $h_1 = 0.005$. Figure 2 shows the values of K for various values of $(\eta_\infty/h_1) \times 10^{-2}$ and J. From this figure we choose the value of K as follows:

Let us assume that we want to take a maximum of 40 points across the boundary layer. If the Reynolds number is not very large, an estimate for the maximum value of $\eta_\infty = 50$ is sufficient. Then taking $h_1 = 0.01$, the ratio of $(\eta_\infty/h_1) \times 10^{-2}$ is 50. Thus from figure 2, $J = 40$, K is approximately 1.19

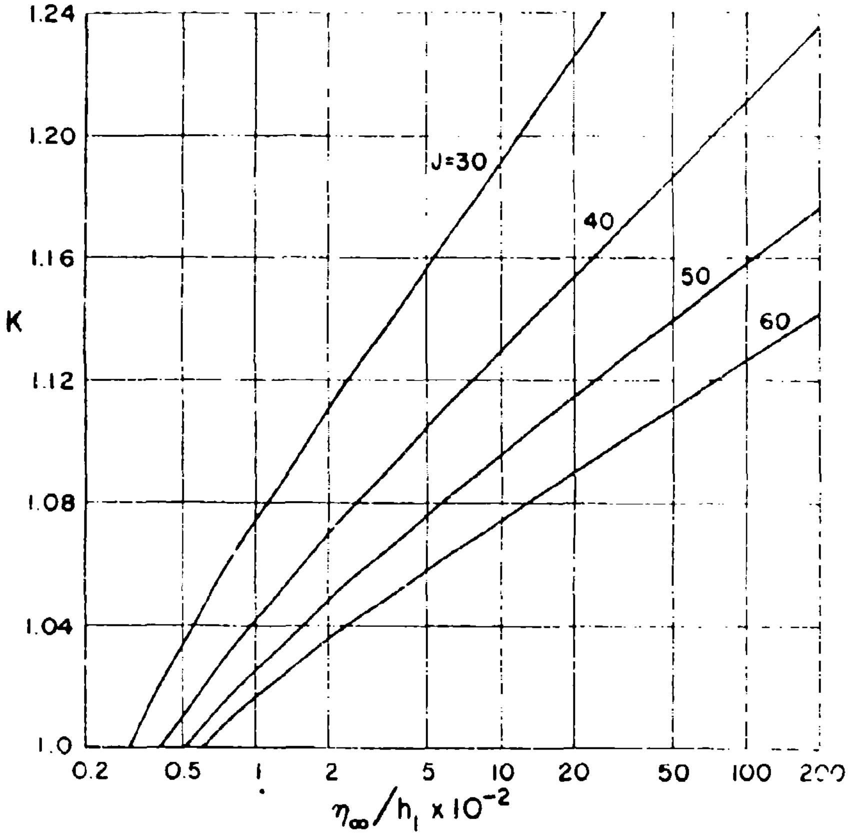
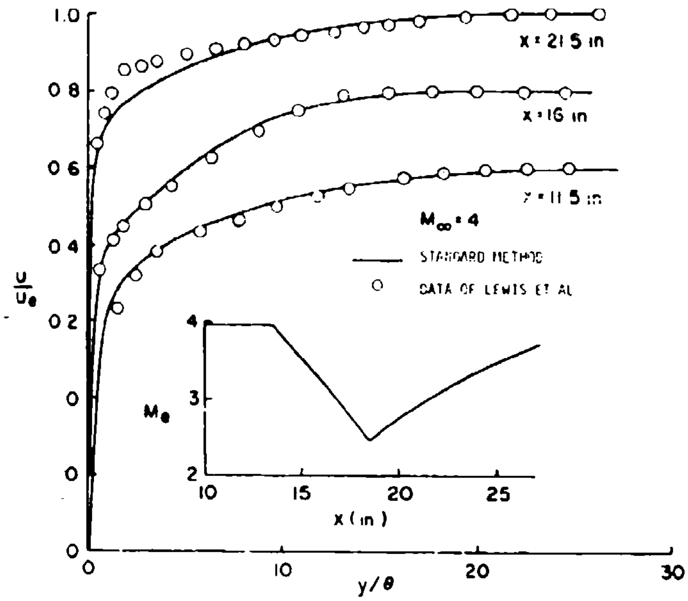


Figure 2. Variable-grid parameter for given step size and boundary-layer thickness.

6.2 Results for the Standard Problem

To test the method for the standard problem we made calculations for the experimental data of Lewis, et al[19], which consists of compressible adiabatic turbulent boundary layers in adverse and favorable pressure gradients. The results are shown in Fig. 3. The calculations were started by matching a zero-pressure-gradient profile (R_θ = 4870) at x = 11.5 in. downstream of the leading edge of the model. Then the experimental Mach-number distribution shown in figure 3a was used to compute the rest of the flow. In general the calculated velocity profiles, local skin-friction and momentum-thickness-Reynolds-number values are in good agreement with experiment. We should point out here that the experimental skin-friction values were obtained by Stanton tube and were not deduced from the experimental velocity profiles a.



b.

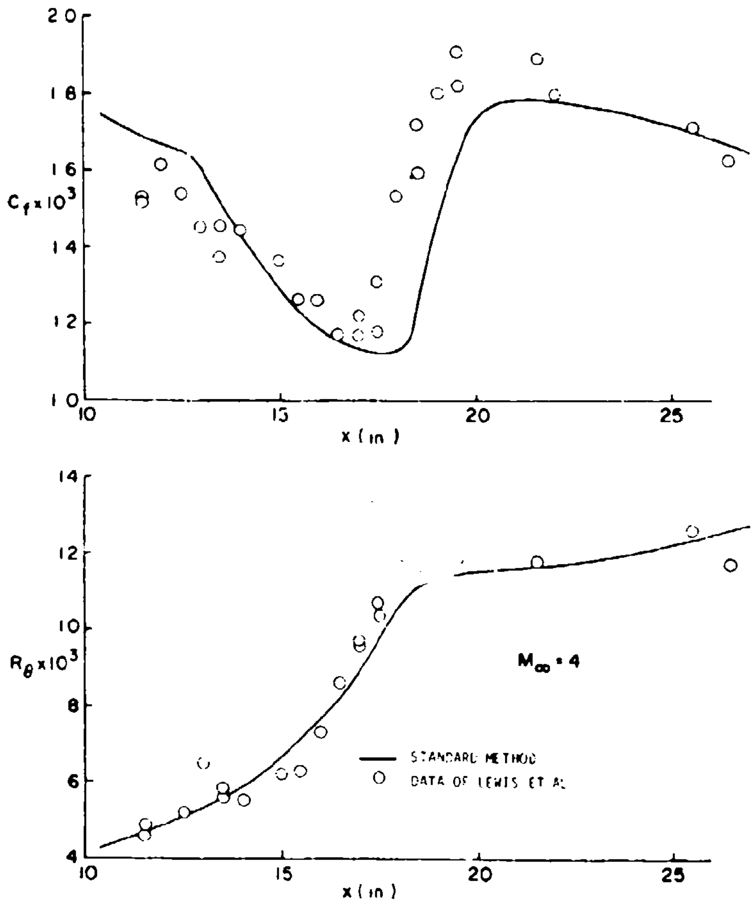


Figure 3. Comparison of calculated and experimental results for the experimental data of Lewis et al. (a) Velocity profiles and external Mach number distribution. (b) Local skin-friction coefficient c_f and Reynolds number R_θ distribution.

6.3 Results for the Inverse Problem

To test our method for the inverse problem for the case of specified c_f, we made calculations for incompressible turbulent boundary layers, and checked the results obtained earlier in another study. We have chosen two experimental incompressible flows from the data reported in the Stanford Conference on Computation of Turbulent Boundary Layers[20]. The flows we have considered are known as 1300, 5200 and 5300 in that conference.

The flow 1300 corresponds to an accelerating flow. The experimental data is due to Ludwieg and Tillmann. Flows 5200 and 5300 correspond to decelerating flows measured by Stratford. They differ from those more common decelerating turbulent flows in that they have a negligible skin friction. Thus, they are on the verge of separating. For this reason it is a very severe test for a numerical method and for exploring the accuracy of the eddy-viscosity formulas. In the 1968 Stanford Conference, of the investigators who used differential methods, only <u>one</u> computed 5200 and <u>none</u> has computed 5300. The accuracy of computing these flows is also important in many design problems as was discussed in the Introduction. The design of the Liebeck airfoils discussed in refs. 5 and 6, for example, is based on the results of boundary layer calculations for a flow with vanishing skin friction.

In making these computations we have first considered the standard problem. That is, for the given experimental velocity distribution and for the given initial velocity profiles at $x = x_0$, we have computed the velocity profiles and the local skin-friction coefficient at each specified x-location. Then we made the calculations for the inverse problem. We specified the computed local skin-friction coefficient as an additional boundary condition at each x-station and computed the velocity distribution by the inverse method. We have thus used the computed skin-friction values, rather than the experimental values, as a boundary condition. Such a procedure is necessary because a slight error in the experimental skin-friction coefficient will severely affect the computed velocity distribution. To discuss this point further, let us consider the data of Stratford, either 5200 or 5300. It may be seen from the skin-friction plots in figures 5b and 6b that the experimental values of c_f show scatter; in an adverse pressure gradient flow, c_f should either stay nearly constant or decrease. If one uses the scattered values as boundary condition and computes the velocity distribution, one would get slight increases and decreases in the velocity distribution with increasing and decreasing c_f, respectively.

The computed results in fig. 4 for the accelerating flow 1300 show very good agreement with experimental data. This indicates that our eddy viscosity formulation is quite satisfactory for this flow. On the other hand, the computed results in figures 5 and 6 for the two decelerating, on-the-verge-of-separating flows, 5200 and 5300, are not satisfactory at all although the computed results at the beginning of these two flows agree well with experiment (see the velocity profiles at $x = 2.9075$ for both 5200 and 5300). This is probably due to the effect of the strong pressure gradient suddenly imposed at $x \geq 3.0$ ft. However, the present method does not break down as almost all other numerical methods do and performs extremely well even under such strong pressure gradients at 5200 and 5300 contain.

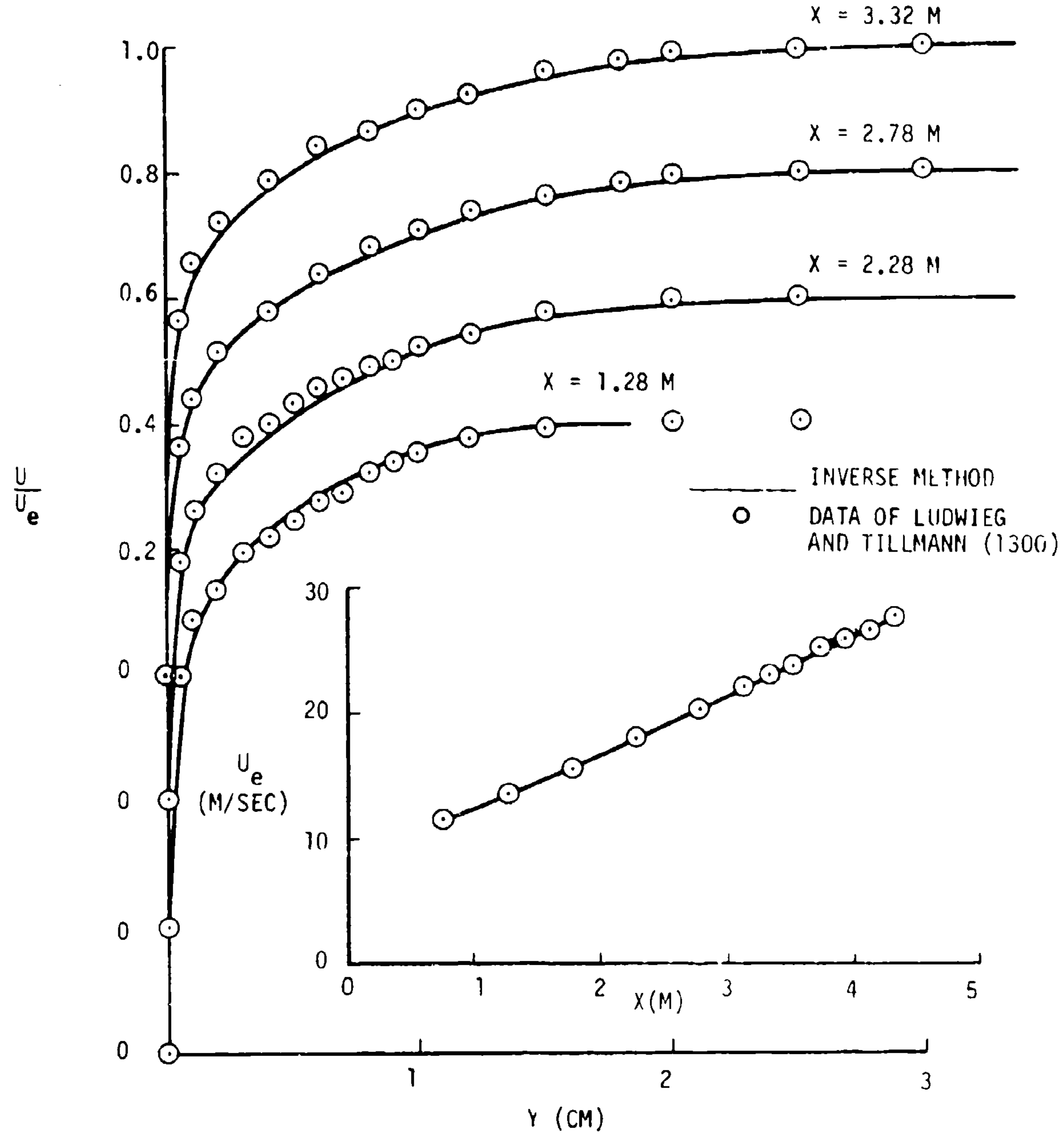


Figure 4. Comparison of calculated and experimental results for the flow 1300.
(a) Velocity profiles and external velocity distribution.

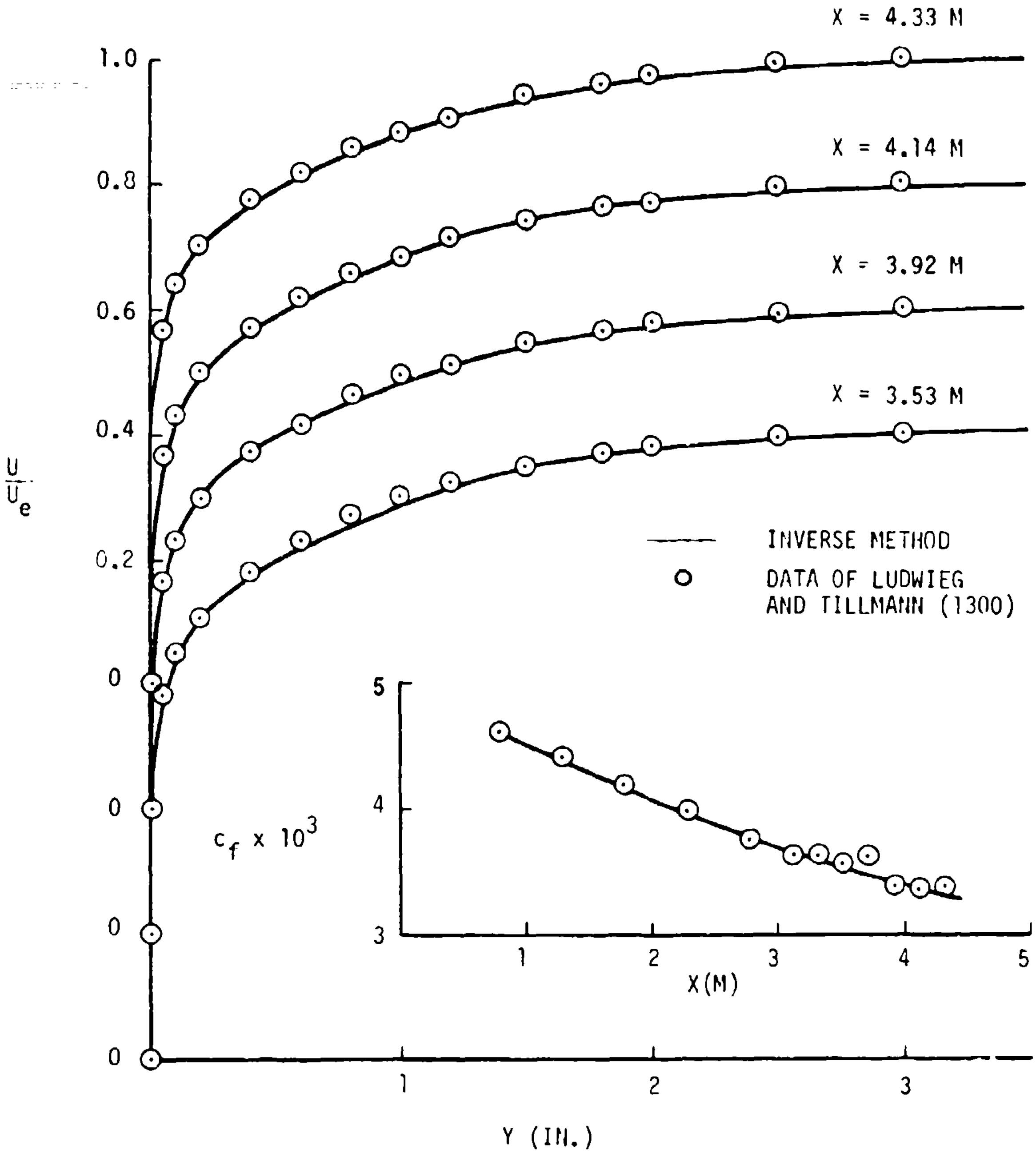


Figure 4. (b) Velocity profiles and local skin-friction-coefficient distribution.

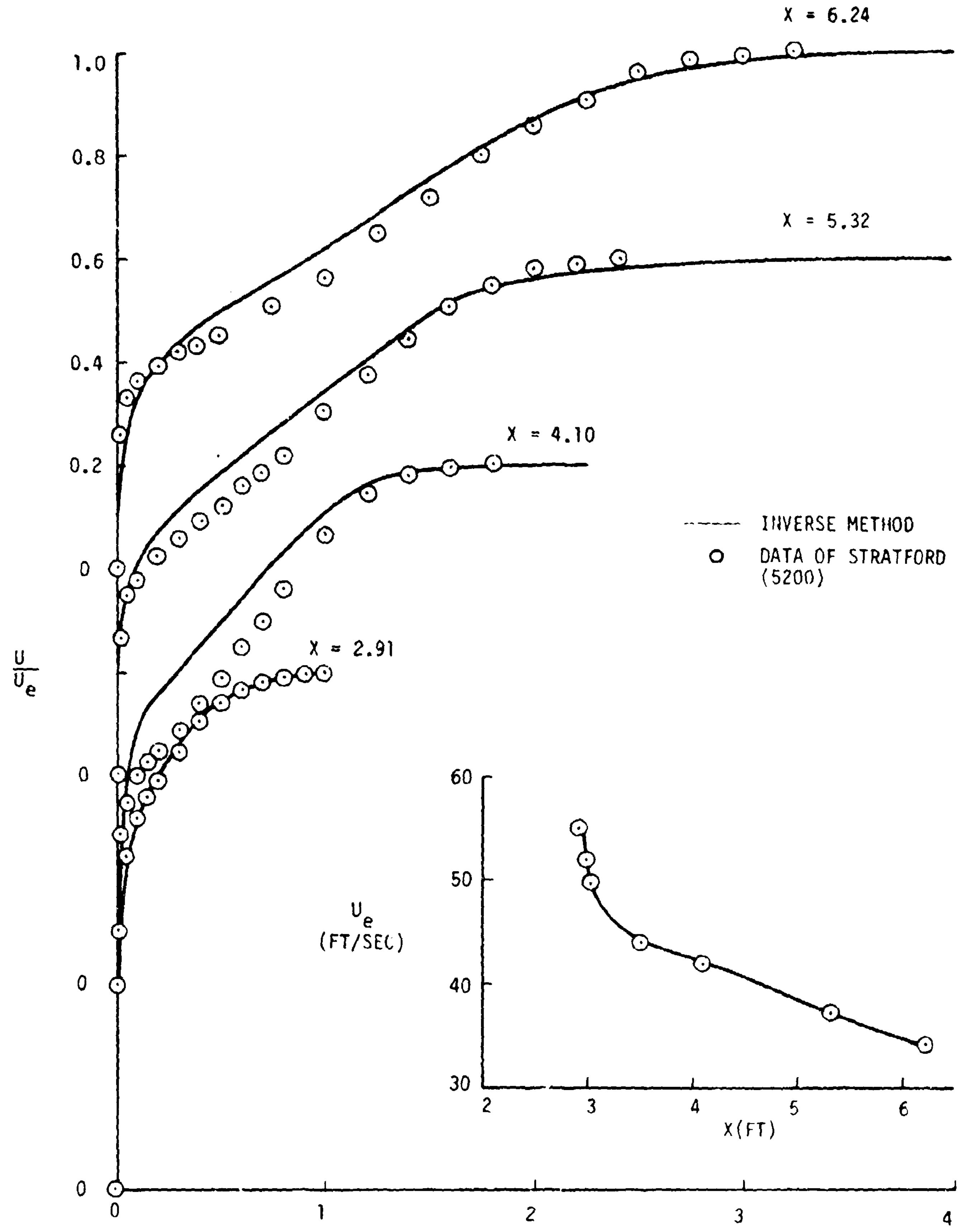


Figure 5. Comparison of calculated and experimental results for the flow 5200. (a) Velocity profiles and external velocity distribution.

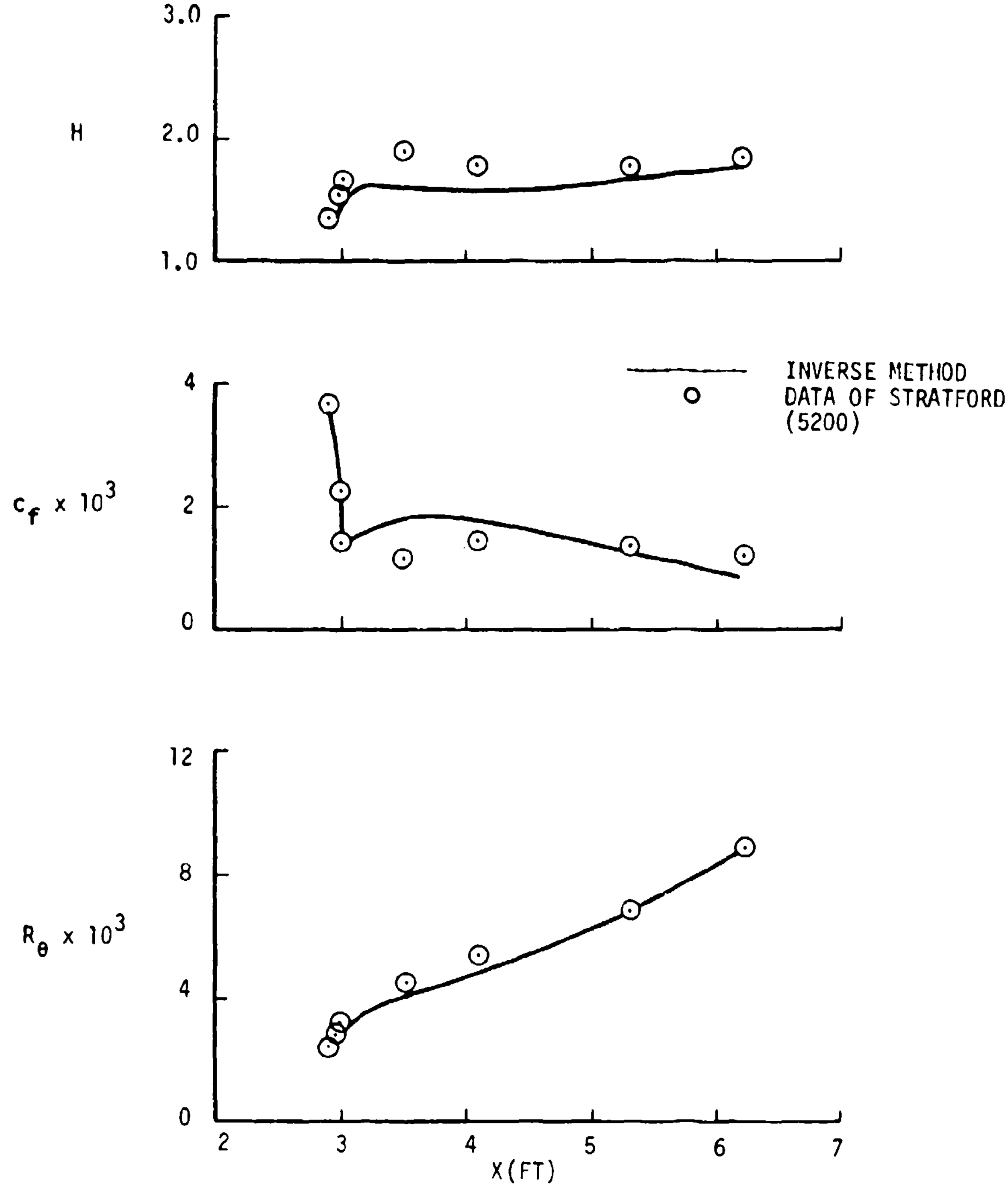


Figure 5. (b) Shape factor H, local skin friction coefficient c_f and Reynolds number, R_θ distribution.

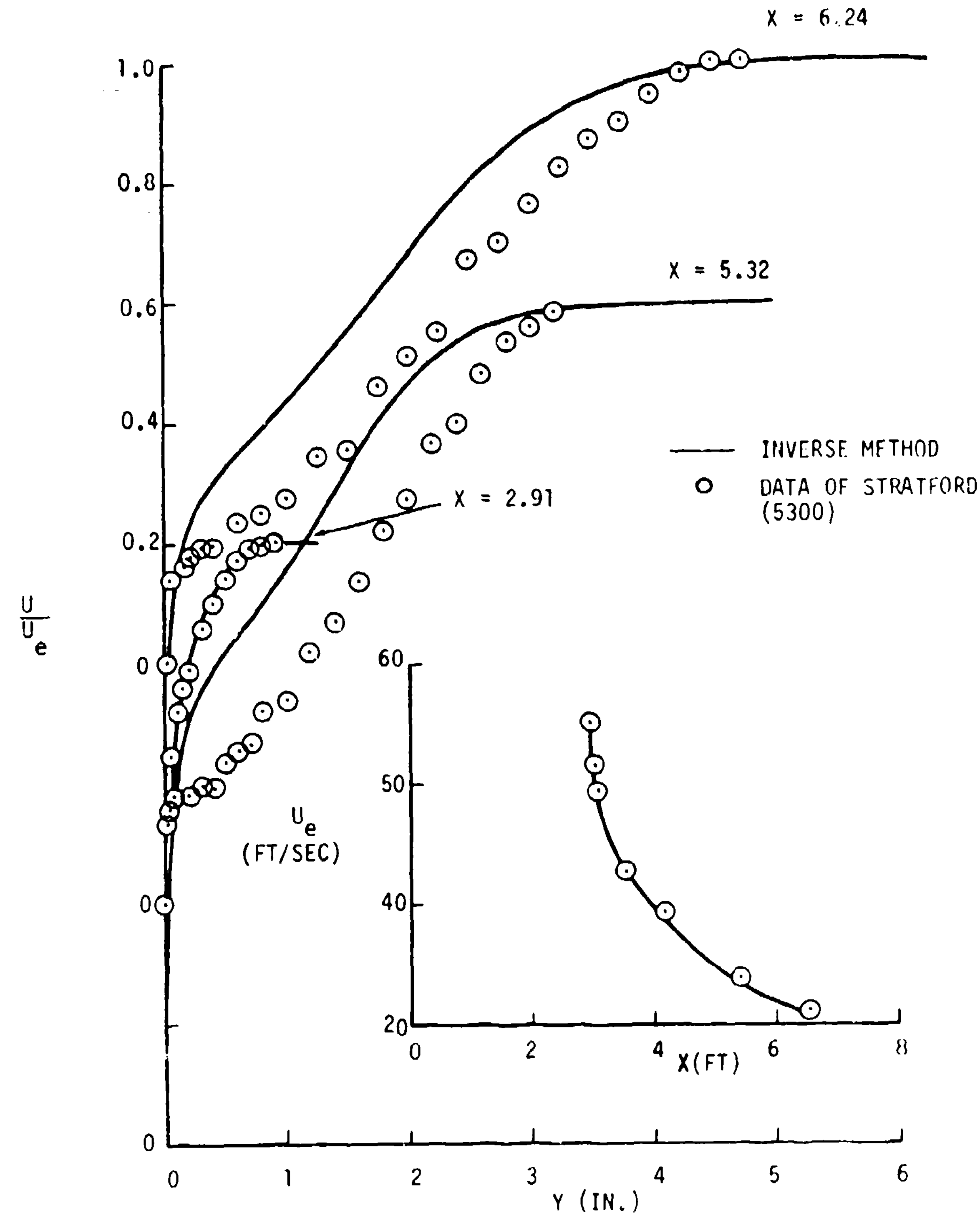


Figure 6. Comparison of calculated and experimental results for the flow 5300.
(a) Velocity profiles and external velocity distribution.

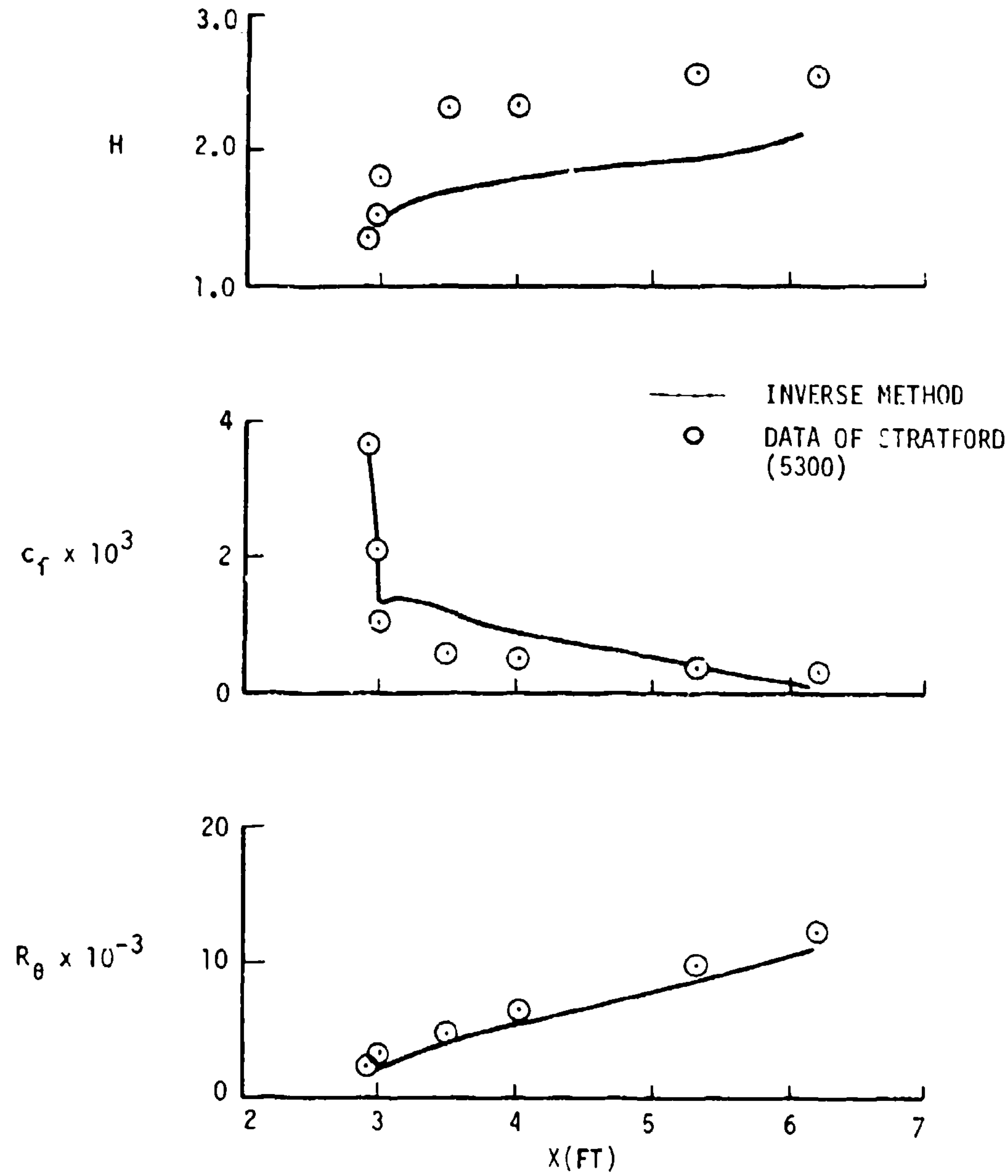


Figure 6. (b) Shape factor H, local skin friction coefficient c_f and Reynolds number, R_θ distribution.

VII. REFERENCES

1. Cebeci, T., Mosinskis, G.J., and Smith, A.M.O.: Calculation of Separation Points in Turbulent Flows. J. of Aircraft, Vol. 9, No. 9, Sept. 1972.

2. Cebeci, T.: Calculation of Three-Dimensional Boundary Layers. I. Swept Infinite Cylinders and Small Cross Flow. AIAA J., Vol. 12, No. 6, June 1974.

3. Cebeci, T., Kaups, K., Mosinskis, G.J. and Rehn, J.A.: Some Problems of the Calculation of Three-Dimensional Boundary-Layer Flows on General Configurations. NASA CR-2285, July 1973.

4. Cebeci, T.: Calculation of Three-Dimensional Boundary Layers. To appear in AIAA J., 1975.

5. Liebeck, R.H. and Ormsbee, A.I.: Optimization of Airfoils for Maximum Lift. J. of Aircraft, Vol. 7, No. 5, Sept.-Oct. 1970.

6. Liebeck, R.H.: A Class of Airfoils Designed for High Lift in Incompressible Flow. J. of Aircraft, Vol. 10, No. 10, Oct. 1973.

7. Catherall, D. and Mangler, K.W.: The Integration of the Two-Dimensional Laminar Boundary-Layer Equations Past the Point of Vanishing Skin Friction. J. of Fluid Mech., Vol. 26, Pt. 1, pp. 163-182, 1966.

8. Keller, H.B. and Cebeci, T.: An Inverse Problem in Boundary-Layer Flows: Numerical Determination of Pressure Gradient for a Given Wall Shear. J. of Computational Physics, Vol. 10, No. 1, Aug. 1972.

9. Klineberg, J.M. and Steger, J.L.: On Laminar Boundary-Layer Separation. AIAA Paper No. 74-94, Jan. 1974.

10. Carter, J.E.: Solutions for Laminar Boundary Layers with Separation and Reattachment. AIAA Paper No. 74-583, June 1974.

11. Briley, W.R.: A Numerical Study of Laminar Separation Bubbles Using the Navier-Stokes Equations. J. of Fluid Mech., Vol. 47, Pt. 4, pp. 713-736, 1971.

12. Cebeci, T., Berkant, N., Silivri, I., and Keller, H.B.: Turbulent Boundary Layers with Assigned Wall Shear. To appear in Computers and Fluids, 1975.

13. Kuhn, G.D. and Nielsen, J.N.: Prediction of Turbulent Separated Boundary Layers. AIAA J., Vol. 12, No. 7, pp. 881-882, July 1974.

14. Cebeci, T., and Smith, A.M.O.: Analysis of Turbulent Boundary Layers. Academic Press, New York, 1974.

15. Keller, H.B.: A New Difference Scheme for Parabolic Problems, in Numerical Solution of Partial Differential Equations. (ed. J. Bramble) Vol. II, Academic Press, New York, 1970.

16. Keller, H.B. and Cebeci, T.: Accurate Numerical Methods for Boundary Layer Flows. II. Two-Dimensional Turbulent Flows. AIAA J., Vol. 10, No. 9, pp. 1193-1200, Sept. 1972.

17. Cebeci, T. and Keller, H.B.: Laminar Boundary Layers with Assigned Wall Shear. Proceedings of the Third International Conf. on Numerical Methods in Fluid Dynamics, Paris, France, July 1972.

18. Isaacson, E. and Keller, H.B.: Analysis of Numerical Methods, p. 58, Wiley, New York, 1966.

19. Lewis, J.E., Gram, R.L. and Kuboto, T.: An Experiment in the Adiabatic Compressible Turbulent Boundary Layer in Adverse and Favorable Pressure Gradients, J. Fluid Mech., Vol. 51, p. 657, 1972.

20. Coles, D.E. and Hirst, E.A.: Proceedings of Computation of Turbulent Boundary Layers, 1968 AFOSR-IFP-Stanford Conf., Vol. 2, Thermosciences Div. 1969, Stanford University, Stanford, California.

Appendix A

BLOCK ELIMINATION METHOD

The linear system (4.2.2), (4.2.6) or (4.2.7) depending on the boundary conditions can be solved in an extremely efficient manner as described by Isaacson and Keller[18] since it has a block tridiagonal structure. This is not obvious and to clarify the solution procedure we write our system in matrix-vector form. There are many ways in which this can be done. They are all equivalent and merely amount to different permutations of the equations or of the unknowns or both. Further, the boundary conditions (4.2.6) or (4.2.7) could be employed to eliminate the five unknowns and thus slightly reduce the order of the system. When the latter is done, the system (4.2.2), and (4.2.6) for the case of specified wall temperature can be written as

$$\mathbb{A}\vec{\delta}_j = \vec{r}_j \tag{A.1}$$

where $\vec{\delta}_j$ and $\vec{r}_j$ are vectors denoted by

$$\vec{\delta}_0 = \begin{bmatrix} \delta v_0 \\ \delta t_0 \\ \delta f_1 \\ \delta v_1 \\ \delta t_1 \end{bmatrix}, \quad \vec{\delta}_j = \begin{bmatrix} \delta u_{j-1} \\ \delta g_{j-1} \\ \delta f_j \\ \delta v_j \\ \delta t_j \end{bmatrix} \; 2 \le j \le J, \qquad \vec{r}_j = \begin{bmatrix} (r_1)_j \\ (r_2)_j \\ (r_3)_j \\ (r_4)_j \\ (r_5)_j \end{bmatrix} \; 1 \le j \le J \tag{A.2}$$

Here the coefficient matrix A is

$$\mathbb{A} = \begin{bmatrix} A_1 & C_1 & & & \\ B_2 & A_2 & C_2 & & \\ & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & \cdot \\ & & B_{J-1} & A_{J-1} & C_{J-1} \\ & & & B_J & A_J \end{bmatrix} \tag{A.3}$$

and A_j, C_j, B_j are 5 x 5 matrices given by $(a_j \equiv h_j/2)$

$$A_1 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ -a_1 & 0 & 0 & -a_1 & 0 \\ 0 & -a_1 & 0 & 0 & -a_1 \\ (\zeta 2)_1 & 0 & (\zeta 3)_1 & (\zeta 1)_1 & 0 \\ 0 & (\beta 2)_1 & (\beta 3)_1 & 0 & (\beta 1)_1 \end{bmatrix}, \quad A_j = \begin{bmatrix} -a_j & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -a_j & 0 \\ 0 & -1 & 0 & 0 & -a_j \\ (\zeta 6)_j & (\zeta 8)_j & (\zeta 3)_j & (\zeta 1)_j & 0 \\ (\beta 8)_j & (\beta 6)_j & (\beta 3)_j & 0 & (\beta 1)_j \end{bmatrix}$$

$$2 \le j \le J \tag{A.4}$$

$$B_j = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -a_j & 0 \\ 0 & 0 & 0 & 0 & -a_j \\ 0 & 0 & (\zeta 4)_j & (\zeta 2)_j & 0 \\ 0 & 0 & (\beta 4)_j & 0 & (\beta 2)_j \end{bmatrix} \; 2 < j \le J \; ; \quad C_j = \begin{bmatrix} -a_j & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ (\zeta 5)_j & (\zeta 7)_j & 0 & 0 & 0 \\ (\beta 7)_j & (\beta 5)_j & 0 & 0 & 0 \end{bmatrix} \; 1 \le j \le J-1$$

The system (A.1) can be solved by the block tridiagonal factorization procedure described by Isaacson and Keller. According to this procedure, we first seek a factorization of the form

$$\mathbf{A} = \mathcal{L}\,\mathcal{U} \tag{A.5}$$

where (with I_j denoting the identity matrices)

$$\mathcal{L} \equiv \begin{bmatrix} \alpha_1 & & & & \\ B_2 & \alpha_2 & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & B_J & \alpha_J \end{bmatrix} \quad \mathcal{U} \equiv \begin{bmatrix} I_1 & \Gamma_1 & & & \\ & I_2 & \Gamma_2 & & \\ & & \cdot & \cdot & \\ & & & \cdot & \cdot \\ & & & & I_J \end{bmatrix} \tag{A.6}$$

From (A.3) and (A.5) it follows that

$$\alpha_1 = A_1, \qquad A_1\Gamma_1 = C_1 \tag{A.7a}$$

$$\alpha_j = A_j - B_j\Gamma_{j-1} \qquad j = 2, 3, \ldots, J \tag{A.7b}$$

$$\alpha_j\Gamma_j = C_J \qquad j = 2, 3, \ldots, J-1 \tag{A.7c}$$

Substituting (A.5) into (A.1) we get

$$\mathcal{L}\mathcal{U}\vec{\delta}_j = \vec{r}_j \tag{A.8}$$

If we let

$$\mathcal{U}\vec{\delta}_j = \vec{w}_j \tag{A.9}$$

Then (A.8) becomes

$$\mathcal{L}\vec{w}_j = \vec{r}_j \tag{A.10}$$

If we denote Γ_j by

$$\Gamma_j = \begin{bmatrix} (\gamma_{11})_j & (\gamma_{12})_j & 0 & 0 & 0 \\ (\gamma_{21})_j & (\gamma_{22})_j & 0 & 0 & 0 \\ (\gamma_{31})_j & (\gamma_{32})_j & 0 & 0 & 0 \\ (\gamma_{41})_j & (\gamma_{42})_j & 0 & 0 & 0 \\ (\gamma_{51})_j & (\gamma_{52})_j & 0 & 0 & 0 \end{bmatrix} \qquad j = 1, 2, 3, \ldots J \tag{A.11}$$

Then from (A.7a) we find that for $j = 1$

$$\gamma_{11} = \frac{1}{(\zeta_2)_1 - (\zeta_1)_1}\left[\frac{(\zeta_1)_1}{a_1} + (\zeta_5)_1 + a_1(\zeta_3)_1\right]$$

$$\gamma_{21} = \frac{1}{(\beta_2)_1 - (\beta_1)_1}\left[(\beta_7)_1 + a_1(\beta_3)_1\right]$$

$$\gamma_{31} = -a_1$$

$$\gamma_{41} = -\frac{1}{a_1} - \gamma_{11} \tag{A.12}$$

$$\gamma_{51} = -\gamma_{21}$$

$$\gamma_{12} = \frac{(\zeta_7)_1}{(\zeta_2)_1 - (\zeta_1)_1}$$

$$\gamma_{22} = \frac{1}{(\beta_2)_1 - (\beta_1)_1}\left[\frac{(\beta_1)_1}{a_1} + (\beta_5)_1\right] \tag{A.12}$$

$$\gamma_{32} = 0$$

$$\gamma_{42} = -\gamma_{12}$$

$$\gamma_{52} = -\frac{1}{a_1} - \gamma_{22}$$

From (A.7b), we can show that for $j \geq 2$, the elements are α_j are:

$$\alpha_j \equiv \begin{bmatrix} (\alpha_{11})_j & (\alpha_{12})_j & 1 & 0 & 0 \\ (\alpha_{21})_j & (\alpha_{22})_j & 0 & -a_j & 0 \\ (\alpha_{31})_j & (\alpha_{32})_j & 0 & 0 & -a_j \\ (\alpha_{41})_j & (\alpha_{42})_j & (\zeta_3)_j & (\zeta_1)_j & 0 \\ (\alpha_{51})_j & (\alpha_{52})_j & (\beta_3)_j & 0 & (\beta_1)_j \end{bmatrix} \tag{A.13}$$

where

$$(\alpha_{11})_j = -a_j + (\gamma_{31})_{j-1}$$

$$(\alpha_{21})_j = -1 + a_j(\gamma_{41})_{j-1}$$

$$(\alpha_{31})_j = a_j(\gamma_{51})_{j-1}$$

$$(\alpha_{41})_j = (\zeta_\epsilon)_j - (\zeta_4)_j(\gamma_{31})_{j-1} - (\zeta_2)_j(\gamma_{41})_{j-1} \tag{A.14}$$

$$(\alpha_{51})_j = (\beta_8)_j - (\beta_4)_j(\gamma_{31})_{j-1} - (\beta_2)_j(\gamma_{51})_{j-1}$$

$$(\alpha_{12})_j = (\gamma_{32})_{j-1}$$

$$(\alpha_{22})_j = a_j(\gamma_{42})_{j-1}$$

$$(\alpha_{32})_j = -1 + a_j(\gamma_{52})_{j-1}$$

$$(\alpha_{42})_j = (\zeta_8)_j - (\zeta_4)_j(\gamma_{32})_{j-1} - (\zeta_2)_j(\gamma_{42})_{j-1}$$

$$(\alpha_{52})_j = (\beta_6)_j - (\beta_4)_j(\gamma_{32})_{j-1} - (\beta_2)_j(\gamma_{52})_{j-1}$$

The unknowns γ_{ik} in (A.11) for $i = 1, 2, 3, 4, 5$ and $k = 1, 2$ are next determined from (A.7c). For $k = 1$, γ_{i1} are:

$$(\gamma_{11})_j = \frac{(c_1)_j(b_2)_j - (b_1)_j(c_2)_j}{\Delta_j}$$

$$(\gamma_{21})_j = \frac{(a_1)_j(c_2)_j - (c_1)_j(a_2)_j}{\Delta_j}$$

$$(\gamma_{31})_j = -a_j - (\alpha_{11})_j(\gamma_{11})_j - (\alpha_{12})_j(\gamma_{21})_j$$

$$(\gamma_{41})_j = [(\alpha_{21})_j(\gamma_{11})_j + (\alpha_{22})_j(\gamma_{21})_j - 1]\, a_j^{-1}$$

$$(\gamma_{51})_j = [(\alpha_{31})_j(\gamma_{11})_j + (\alpha_{32})_j(\gamma_{22})_j]\, a_j^{-1}$$

$$(a_1)_k = (\alpha_{41})_j - (\zeta_3)_j(\alpha_{11})_j + (\zeta_1)_j(\alpha_{21})_j a_j^{-1}$$

$$(b_1)_j = (\alpha_{42})_j - (\zeta_3)_j(\alpha_{12})_j + (\zeta_1)_j(\alpha_{22})_j a_j^{-1} \qquad \text{(A.15a)}$$

$$(c_1)_j = (\zeta_5)_j + (\zeta_3)_j a_j + (\zeta_1)_j a_j^{-1}$$

$$(a_2)_j = (\alpha_{51})_j - (\beta_3)_j(\alpha_{11})_j + (\beta_1)_j(\alpha_{31})_j a_j^{-1}$$

$$(b_2)_j = (\alpha_{52})_j - (\beta_3)_j(\alpha_{12})_j + (\beta_1)_j(\alpha_{32})_j a_j^{-1}$$

$$(c_2)_j = (\beta_7)_j + (\beta_3)_j a_j$$

$$\Delta_j = (a_1)_j(b_2)_j - (b_1)_j(a_2)_j$$

For $k = 2$, γ_{i2} are:

$$(\gamma_{12})_j = \frac{-(b_1)_j(c_2)_j}{\Delta_j}$$

$$(\gamma_{22})_j = \frac{(a_1)_j(c_2)_j}{\Delta_j}$$

$$\text{(A.15b)}$$

$$(\gamma_{32})_j = -(\alpha_{11})_j(\gamma_{12})_j - (\alpha_{12})_j(\gamma_{22})_j$$

$$(\gamma_{42})_j = [(\alpha_{21})_j(\gamma_{12})_j + (\alpha_{22})_j(\gamma_{22})_j]\, a_j^{-1}$$

$$(\gamma_{52})_j = [(\alpha_{31})_j(\gamma_{12})_j + (\alpha_{32})_j(\gamma_{22})_j - 1]\, a_j^{-1}$$

where $(a_1)_j$, $(b_1)_j$, $(a_2)_j$, $(b_2)_j$ and Δ_j are the same as in (A.15a); $(c_2)_j$ is different and is given by

$$(c_2)_j = (\beta_5)_j + (\beta_1)_j a_j^{-1}$$

To determine $\vec{w}_j$ we use (A.10) and the definition of $\mathcal{L}$ in (A.6). If we denote $\vec{w}_j$ by

$$\vec{w}_j = \begin{bmatrix} (w_1)_j \\ (w_2)_j \\ (w_3)_j \\ (w_4)_j \\ (w_5)_j \end{bmatrix} \tag{A.16}$$

then it can be shown that

$$\alpha_1 \vec{w}_1 = \vec{r}_1 \tag{A.16a}$$

$$\alpha_j \vec{w}_j = \vec{r}_j - B_j \vec{w}_{j-1} \qquad 2 \le j \le J \tag{A.16b}$$

Thus for $j = 1$, using the definitions of α_1, $\vec{w}_1$ and $\vec{r}_1$ from (A.16a) we can write

Thus for $j = 1$, using the definitions of α_1, $\vec{w}_1$ and $\vec{r}_1$ from (A.16a) we can write

$$
\begin{aligned}
(w_1)_j &= [(r_2)_j(\zeta_1)_j a_j^{-1} + (r_4)_j - (\zeta_3)_j(r_1)_j]\,[(\zeta_2)_j - (\zeta_1)_j]^{-1} \\
(w_2)_j &= [(r_3)_j(\beta_2)_j a_j^{-1} + (r_5)_j - (\beta_3)_j(r_1)_j]\,[(\beta_2)_j - (\beta_1)_j]^{-1} \\
(w_3)_j &= (r_1)_j \\
(w_4)_j &= -w_1 - (r_2)_1 a_1^{-1} \\
(w_5)_j &= -w_2 - (r_3)_1 a_1^{-1}
\end{aligned}
\tag{A.17}
$$

For $2 \le j \le J$, we use (A.16b). If we denote its right-hand side by $\vec{n}$ with

$$
\vec{n} = \begin{bmatrix} (n_1)_j \\ (n_2)_j \\ (n_3)_j \\ (n_4)_j \\ (n_5)_j \end{bmatrix}
$$

then

$$
\begin{aligned}
(w_1)_j &= \frac{(c_1)_j(b_2)_j - (b_1)_j(c_2)_j}{\Delta_j} \\
(w_2)_j &= \frac{(a_1)_j(c_2)_j - (c_1)_j(a_2)_j}{\Delta_j} \\
(w_3)_j &= (n_1)_j - (\alpha_{11})_j(w_1)_j - (\alpha_{12})_j(w_2)_j \\
(w_4)_j &= [(\alpha_{21})_j(w_1)_j + (\alpha_{22})_j(w_2)_j - (n_2)_j]a_j^{-1} \\
(w_5)_j &= [(\alpha_{31})_j(w_1)_j + (\alpha_{32})_j(w_2)_j - (n_3)_j]a_j^{-1}
\end{aligned}
\tag{A.18}
$$

where $(a_1)_j$, $(b_1)_j$, $(a_2)_j$, $(b_2)_j$, Δ_j are given by the definitions in (A.15a) and $\vec{n}_j$, $(c_1)_j$ and $(c_2)_j$ are:

$$(n_1)_j = (r_1)_j + (w_3)_{j-1}$$

$$(n_2)_j = (r_2)_j + a_j(w_4)_{j-1}$$

$$(n_3)_j = (r_3)_j + a_j(w_5)_{j-1}$$

$$(n_4)_j = (r_4)_j - [(\zeta_4)_j(w_3)_{j-1} + (\zeta_2)_j(w_4)_{j-1}]$$

$$(n_5)_j = (r_5)_j - [(\beta_4)_j(w_3)_{j-1} + (\beta_2)_j(w_5)_{j-1}]$$

$$(c_1)_j = (n_4)_j - (\zeta_3)_j(n_1)_j + (\zeta_1)_j(n_2)_j a_j^{-1}$$

$$(c_2)_j = (n_5)_j - (\beta_3)_j(n_1)_j + (\beta_1)_j(n_3)_j a_j^{-1}$$

Finally the perturbation quantities denoted by $\vec{\delta}_j$ are calculated from (A.6) and (A.9). It follows that

$$\vec{\delta}_J = \vec{w}_J \tag{A.20a}$$

$$\vec{\delta}_j = \vec{w}_j - \Gamma_j \vec{\delta}_{j+1} \qquad 2 \le j \le J-1 \tag{A.20b}$$

Thus at $j = J$,

$$\begin{aligned} \delta u_{J-1} &= (w_1)_J \\ \delta g_{J-1} &= (w_2)_J \\ \delta f_J &= (w_3)_J \\ \delta v_J &= (w_4)_J \\ \delta t_J &= (w_5)_J \end{aligned} \tag{A.21}$$

For $2 \le j \le J-1$,

$$\begin{aligned} \delta u_{j-1} &= (w_1)_j - (\gamma_{11})_j \delta u_j - (\gamma_{12})_j \delta g_j \\ \delta g_{j-1} &= (w_2)_j - (\gamma_{21})_j \delta u_j - (\gamma_{22})_j \delta g_j \\ \delta f_j &= (w_3)_j - (\gamma_{31})_j \delta u_j - (\gamma_{32})_j \delta g_j \\ \delta v_j &= (w_4)_j - (\gamma_{41})_j \delta u_j - (\gamma_{42})_j \delta g_j \\ \delta t_j &= (w_5)_j - (\gamma_{51})_j \delta u_j - (\gamma_{52})_j \delta g_j \end{aligned} \tag{A.22}$$

For $j = 1$

$$
\begin{aligned}
\delta v_0 &= (w_1)_1 - (\gamma_{11})_1 \delta u_1 - (\gamma_{12})_1 \delta g_1 \\
\delta g_0 &= (w_2)_1 - (\gamma_{21})_1 \delta u_1 - (\gamma_{22})_1 \delta g_1 \\
\delta f_1 &= (w_3)_1 - (\gamma_{31})_1 \delta u_1 - (\gamma_{32})_1 \delta g_1 \\
\delta v_1 &= (w_4)_1 - (\gamma_{41})_1 \delta u_1 - (\gamma_{42})_1 \delta g_1 \\
\delta t_1 &= (w_5)_1 - (\gamma_{51})_1 \delta u_1 - (\gamma_{52})_1 \delta g_1
\end{aligned}
\qquad (A.23)
$$

Equations starting from (A.2) to (A.23) are for the case of specified wall temperature. When the wall heat flux is specified some of those equations in (A.2) to (A.23) change. For example $\vec{\delta}_0$ and A_1 in (A.2) and (A.4) become

$$
\vec{\delta}_0 = \begin{bmatrix} \delta v_0 \\ \delta g_0 \\ \delta f_1 \\ \delta v_1 \\ t_1 \end{bmatrix} , \qquad A_1 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ -a_1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -a_1 & -a_1 \\ (\zeta_2)_1 & (\zeta_8)_1 & (\zeta_3)_1 & (\zeta_1)_1 & 0 \\ 0 & (\beta_6)_1 & (\beta_3)_1 & 0 & (\beta_1)_1 \end{bmatrix} \qquad (A.24)
$$

These new definitions lead to the new equations for γ_{ik} for $j = 1$, and for $(w_i)_1$ with $i = 1, 2, \ldots, 5$ and $k = 1, 2$. The equations for γ_{ik} are:

$$
\begin{aligned}
(\gamma_{31})_1 &= -a_1 \\
(\gamma_{21})_1 &= \frac{1}{(\beta_6)_1 - (\beta_1)_1 a_1^{-1}} \left[(\beta_7)_1 - (\beta_3)_1 (\gamma_{31})_1 \right] \\
(\gamma_{11})_1 &= \frac{1}{(\zeta_2)_1 - (\zeta_1)_1} \left[(\zeta_5)_1 - (\zeta_3)_1 (\gamma_{31})_1 + \frac{(\zeta_1)_1}{a_1} - (\zeta_8)_1 (\gamma_{21})_1 \right] \\
(\gamma_{41})_1 &= -(\gamma_{11})_1 - a_1^{-1} \\
(\gamma_{51})_1 &= -(\gamma_{21})_1 a_1^{-1} \\
(\gamma_{32})_1 &= 0
\end{aligned}
\qquad (A.25)
$$

$$(\gamma_{22})_1 = \frac{1}{(\beta_6)_1 - (\beta_1)_1 a_1^{-1}} [(\beta_5)_1 + (\beta_1)_1 a_1^{-1}]$$

$$(\gamma_{52})_1 = -[1 + (\gamma_{22})_1)]a_1^{-1}$$

$$(\gamma_{42})_1 = \frac{1}{(\zeta_2)_1 - (\zeta_1)_1} [(\zeta_7)_1 - (\zeta_8)_1(\gamma_{22})_1]$$

$$(\gamma_{12})_1 = -(\gamma_{42})_1$$

The equations for $(w_i)_1$ are:

$$(w_3)_1 = (r_1)_1$$

$$(w_2)_1 = \frac{1}{(\beta_6)_1 - (\beta_1)_1 a_1^{-1}} [(r_5)_1 - (\beta_3)_1(w_3)_1 + (\beta_1)_1(r_3)_1 a_1^{-1}]$$

$$(w_1)_1 = \frac{1}{(\zeta_2)_1 - (\zeta_1)_1} [(r_4)_1 - (\zeta_3)_1(w_3)_1 + (\zeta_1)_1(r_2)_1 a_1^{-1} - (\zeta_8)_1(w_2)_1]$$

$$(w_4)_1 = -(w_1)_1 - (r_2)_1 a_1^{-1} \tag{A.26}$$

$$(w_5)_1 = -[(w_2)_1 + (r_3)_1]a_1^{-1}$$

The rest are the same except for the second equation in (A.23). That equation is replaced by

$$\delta g_0 = (w_2)_1 - (\gamma_{21})_1 \delta u_1 - (\gamma_{22})_1 \delta g_1 \tag{A.27}$$